Relation of orbital integrals on $SO(5)$ and $PGL(2)$.

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Abstract

We relate the "Fourier" orbital integrals of corresponding spherical functions on the p-adic groups $SO(5)$ and $PGL(2)$. The correspondence is defined by a "lifting" of representations of these groups. This is a local "fundamental lemma" needed to compare the geometric sides of the global Fourier summation formulae (or relative trace formulae) on these two groups. This comparison leads to conclusions about a well known lifting of representations from $PGL(2)$ to $PGSp(4)$. This lifting produces counter examples to the Ramanujan conjecture.

Introduction. Let G be the special orthogonal group, defined over a local field F, by an anti-diagonal form, with upper triangular minimal parabolic subgroup. An explicit definition is given in Section 0, where G is denoted by $SO(3,2)$. Let C_θ ($\theta \in F^\times$) be a subgroup of G isomorphic to the special orthogonal groups $SO(2,2)$ or $SO(3,1)$ (see Section 0). Denote by P the maximal upper triangular parabolic subgroup of G with abelian unipotent radical N. For any spherical function $f \in C_c(K\backslash G/K)$, K being the standard maximal compact subgroup of G, consider the Fourier orbital integral

$$\int_N \int_{C_\theta} f(ngh)\psi_N(n)dhdn,$$

where ψ_N is a certain character on N depending on a fixed character ψ of F with conductor R (integers of F). The N-C_θ-orbits of maximal dimension are of the form $Na_\alpha\gamma_0 C_\theta$, where a_α is the diagonal matrix $\mathrm{diag}(\alpha, 1, 1, 1, \alpha^{-1})$ and γ_0 is defined below. We are interested in g of the form $a_\alpha\gamma_0$, and denote the integral for such g by $\Psi(\alpha, f)$.

Let H be the group $PGL(2)$ over F. By the Bruhat decomposition it is $N'A' \cup N'wN'A'$. Define a character on the upper unipotent subgroup N' by $\psi_{N'}(n') = \psi(x)$, where $n' = n'(x) \in N'$.

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For any spherical function f' on H (i.e. $f \in C_c(K'\backslash H/K')$), define the Fourier orbital integral

$$\Psi'(\alpha, f') = \int_{N'} \int_{A'} f'(n'wn'(\alpha)a')\psi_{N'}(n')\iota(a')da'dn',$$

where ι is χ_0 (the unramified quadratic character on $F^\times$) in the split case, and 1 in the non-split case.

Let $\pi_\zeta = I_G(\zeta, 1/2 + \zeta)$ be a (certain) unramified representation of G, induced from its Borel subgroup B. Let $\pi'_\zeta = I_H(\zeta, -\zeta)$ be a (certain) unramified representation of H, induced from its Borel subgroup B'. We say that two spherical functions f on G and f' on H are corresponding if their Satake transforms are equal, i.e. tr $\pi_\zeta(f) =$ tr $\pi'_\zeta(f')$, for all complex numbers ζ.

Consider a pair of corresponding functions (f, f'). The main result of this paper shows that their Fourier orbital integrals are related by

$$\Psi(\alpha, f) = (\theta, \alpha)\psi(\alpha)|\alpha|\Psi'(\alpha^{-1}, f'),$$

where (θ, α) is a Hilbert symbol.

Our paper was motivated by Flicker-Mars [FM], which deals with lifting representations from $\mathbf{H} = PGL(2)$ to $\mathbf{G} = PGSp(4)$. It uses the property that the lifts have periods with respect to the cycle (closed subgroup) $\mathbf{C}_\theta$. Here $\mathbf{C}_\theta$ is the centralizer of $\mathrm{diag}(a_\theta, a_\theta)$ ($a_\theta = \mathrm{antidiag}(1, \theta)$) in $\mathbf{G}$. This property led [FM] to apply the theory of the Fourier summation formula for $PGSp(4)$ and the cycle $\mathbf{C}_\theta$ over a global field. This summation formula is a special case of Jacquet's relative trace formula. Other approaches to establishing this lifting of representations use the theory of the Weil representation (see Oda [O], Rallis-Schiffmann [R], [RS], Langlands [L], Piatetski-Shapiro [PS]).

The Fourier summation formula is obtained by integrating the kernel $K_f(n, h)$, in fact its product $K_f(n, h)\overline{\psi}_N(n)$ with the complex conjugate of the value of the character ψ_N on $\mathbf{N}(\mathbb{A})$, over $n \in \mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})$ and $h \in \mathbf{C}_\theta(F)\backslash\mathbf{C}_\theta(\mathbb{A})$. This kernel of the convolution operator $r(f)$ on the space of cusp forms on $\mathbf{G}(\mathbb{A})$, has the geometric expansion $\sum_{\gamma\in\mathbf{G}(F)} f(n^{-1}\gamma h)$, and a spectral expansion. One compares the geometric side of the summation formula on $\mathbf{G}(\mathbb{A})$ $(= PGSp(4, \mathbb{A}))$ to the geometric side of a similar summation formula on $\mathbf{H}(\mathbb{A})$ $(= PGL(2, \mathbb{A}))$. The equality of the geometric sides (for different test functions) implies the equality of the spectral sides of these two formulae. This can be used to obtain various conclusions about lifting of representations from $PGL(2, \mathbb{A})$ to $PGSp(4, \mathbb{A})$. To carry out the separation argument which plays a key role in these studies one needs a "fundamental lemma", which asserts that corresponding spherical functions on $PGSp(4)$ and $PGL(2)$ have matching Fourier orbital integrals.

Proposition 8 of [FM] states a precise form of the conjectured fundamental lemma. A direct proof of this statement for the unit elements in the Hecke algebras is given in Proposition 6 of [FM].

Note that under the isomorphism of $PGSp(4)$ with $SO(3, 2)$, if θ is a square in $F^\times$, the image of $\mathbf{C}_\theta$ is the split group $SO(2, 2)$, and if θ is non-square, it is $SO(3, 1)$. This paper proves the fundamental lemma conjectured in [FM] for the pairs of local groups

$SO(3,2)/SO(3,1)$ and $SO(3,2)/SO(2,2)$; remarkable cancellations simplify the proof, indeed make it possible.

The proof is based on computing the Fourier transforms of the orbital integrals (referred to also as the Mellin transform, since the variable of integration is multiplicative). By the Fourier inversion formula, the equality of the Fourier transforms of the orbital integrals implies the equality of the orbital integrals themselves. This approach avoids dealing with the asymptotic behaviour of our orbital integrals. It was first used in Jacquet [J] for the unit element, and then extended by Mao [M] for the general elements (in their case of $GL(3)$). In our case, the unit element is treated in [FM] by direct computations. Here, we give the complete proof of the "fundamental lemma" of [FM]. Another interesting question in this direction is the generalization of these results to the case of $SO(n)/SO(n-1)$. We hope that this method would apply in this case too.

Our "Fourier" situation is significantly different from that of standard conjugacy, where it is known that: (1) the fundamental lemma for the unit element implies it for general spherical functions, and (2) the knowledge of the transfer of orbital integrals of general functions implies in principle the fundamental lemma for the unit element, and vice versa. No such results are known in the "Fourier" setting of this paper, in particular since there is no analogue of the reduction of orbital integrals to ones of elliptic elements on Levi factors. Of course it will be of much interest to establish analogous of (1) and (2) in the "Fourier" case.

Professor Y. Flicker suggested this problem to me. I would like to thank him for numerous helpful discussions on this subject, and Dr. Z. Mao for discussions about his paper.

0. Statement of results. Let F be a local non-archimedean field, of residual characteristic $\neq 2$. Denote by R the (local) ring of integers of F. Let $\boldsymbol{\pi} \in F$ be a generator of the maximal ideal of R. Denote by q the number of elements of the residue field $\mathbb{F} = R/\boldsymbol{\pi}R$ of R. Normalize the absolute value on $F^\times$ by $|\boldsymbol{\pi}| = q^{-1}$. Fix an additive character ψ on F with conductor R (i.e. ψ is trivial on R but not on $\boldsymbol{\pi}^{-1}R$).

Let $G = SO(3,2;F)$ be the the group of $g \in SL(5;F)$ with ${}^tgJg = J$, where ${}^tg$ is the transpose of g and $J = J_5$. Here $J_n = (\delta_{n+1-i,i})$ is the n by n matrix with 1's on the antidiagonal and 0's everywhere else. Then G is the split special orthogonal group in five variables. Denote by V the five dimensional vector space of columns, over F. The group G acts on V via multiplication on the left. In the split case, let $v_0 \in V$ be the column ${}^t(0,0,1,0,0)$. Set $C = \mathrm{Stab}_G(v_0)$. Then C is the split special orthogonal group over F in 4 variables; we denote it by $SO(2,2;F)$. The symmetric space G/C is known to be isomorphic (via the map $g \mapsto gv_0$) to a four dimensional closed subvariety S of V, given by a quadratic equation. In the non-split case, let $v_0 \in V$ be the column ${}^t(0,2\theta,0,1,0)$, θ not in $(F^\times)^2$. In Section I.2 we define a subgroup C'_θ of $PGSp(4)$. We denote by $C_\theta = SO(3,1;F)$ the image of C'_θ under the isomorphism from $PGSp(4)$ to $SO(3,2)$. In Lemma 1.4 we show that $C_\theta = \mathrm{Stab}_G(v_0)$. The quotient G/C_θ is known to be isomorphic (via the map $g \mapsto gv_0$) to a four dimensional closed subvariety S of V,

given by a quadratic equation. To simplify the notations, we write C_θ for both split and non-split cases. The split case corresponds to C_1, where $C_1 = C$.

Denote by P the maximal upper triangular parabolic subgroup of G with abelian unipotent radical, N. The subgroup B denotes the upper triangular Borel subgroup of G. Let $K = SO(3,2;R)$ be the standard maximal compact subgroup of G. Define

$$A = \{a_\alpha = \mathrm{diag}(\alpha, 1, 1, 1, \alpha^{-1});\ \alpha \in F^\times\},\ M = \{\mathrm{diag}(1, m, 1);\ m \in SO(J_3)\}.$$

Then $P = NMA$. Let A_0 be the diagonal subgroup of G and N_0 the maximal unipotent subgroup of B. We have $B = A_0 N_0$. Note that N is a subgroup of N_0, isomorphic to F^3. We will write $n = n(x_1, x_2, x_3)$ (as in I.1). In the split case, define a character ψ_N on N by $\psi_N(n) = \psi(x_2)$, where $n = n(x_1, x_2, x_3)$. In the non-split case, set $\psi_{N,\theta}(n) = \psi(x_1 + 2\theta x_3)$.

The subgroup N acts on G/C_θ by multiplication on the left, turning it into a disjoint union of N-orbits. By Proposition 1.5 of Section I the N-C_θ-double cosets in G of maximal dimension (which is equal to 9, as $\dim(N) = 3$ and $\dim(C_\theta) = 6$) are represented by $a_\alpha \gamma_0$, $a_\alpha \in A$, for a certain matrix γ_0, defined at the beginning of Section I. Moreover the N-orbits of S of maximal dimension, 3, are of the form $N a_\alpha \gamma_0 v_0$, $a_\alpha \in A$.

As usual, denote by $C(X)$ the space of complex valued functions on an l-space X (see [BZ]), and in $C_c^\infty(X)$, the subscript "c" indicates "compactly supported", and "∞" means "locally constant". For any $f \in C_c^\infty(G)$, define

$$\phi_f(gv_0) = \int_C f(gh)dh.$$

Then $\phi_f \in C_c^\infty(S)$. If f is a spherical function (i.e. K-biinvariant, or $f \in C_c(K\backslash G/K)$), or even if only $f \in C_c(K\backslash G)$, then $\phi_f \in C_c(K\backslash S)$. For any $\phi \in C_c^\infty(S)$, define the orbital integral

$$\Psi(\alpha, \phi) = \int_N \phi(n a_\alpha \gamma_0 v_0)\psi_N(n)dn.$$

Let H be the group $PGL(2, F)$. Its elements will be denoted by their representatives in $GL(2)$. Note that H has a trivial center. Denote by B' the upper triangular Borel subgroup of H. Let $K' = PGL(2, R)$ be the standard maximal compact subgroup of H. We have $B' = N'A'$, where

$$N' = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix};\ x \in F \right\},\ A' = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix};\ \alpha \in F^\times \right\}.$$

Define a character $\psi_{N'}$ of N' by $\psi_{N'}(n(x)) = \psi(x)$. Let χ_0 be the trivial or an unramified quadratic character of $F^\times$ (i.e. $\chi_0(\boldsymbol{\pi})^2 = 1$, and $\chi_0(R^\times) = 1$). Put $\iota = \chi_0$ in the split case and 1 in the non-split case. Define the integral

$$\Psi_H(\alpha, f') = \int_{N'} \int_{A'} f' \left(n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \iota(a) \psi_{N'}(n) dn d^\times a.$$

Denote by $C_c(N'\backslash H, \psi_{N'})$ the space of complex valued compactly supported modulo N' functions ϕ' on H, which satisfy (for any $n \in N'$) the relation

$$\phi'(ng) = \overline{\psi}_{N'}(n)\phi'(g),$$

where $\overline{z}$ denotes the complex conjugate of z. Write $C_c(N'\backslash H/K', \psi_{N'})$ for the space of such right K'-invariant functions. Given $f' \in C_c^\infty(H)$, define a function $\phi'_{f'} \in C_c(N'\backslash H, \psi_{N'})$ on H by

$$\phi'_{f'}(g) = \int_{N'} \psi_{N'}(n) f'(ng) dn.$$

If f' is K'-biinvariant, then $\phi'_{f'} \in C_c(N'\backslash H/K', \psi_{N'})$. Define the integral

$$\Psi'(\alpha, \phi'_{f'}) = \int_{A'} \phi'_{f'}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\iota(a) d^\times a.$$

Thus, by definition $\Psi'(\alpha, \phi'_{f'}) = \Psi_H(\alpha, f')$.

Definition. The functions $f \in C_c^\infty(G)$ and $f' \in C_c^\infty(H)$ are called *matching* if for every $\alpha \in F^\times$ we have

$$\Psi(\alpha, \phi_f) = (\theta, \alpha)\psi(\alpha)|\alpha|\Psi'(\alpha^{-1}, \phi'_{f'}).$$

Note that in the split case $(\theta, \alpha) = 1$.

Let $\pi = I_G(\zeta, \zeta')$ be the representation of the group G which is normalizedly induced from the character $|\alpha_1|^\zeta |\alpha_2|^{\zeta'}$ of the Borel subgroup, B, where α_1 and α_2 are the two simple roots of G with respect to B. The space of this representation consists of locally constant functions ϕ, such that

$$\phi(nak) = \delta_B^{1/2}(a)|\alpha_1(a)|^\zeta |\alpha_2(a)|^{\zeta'}\phi(k),$$

where $n \in N$, $a = \mathrm{diag}(\alpha, \beta, 1, \beta^{-1}, \alpha^{-1})$ and $\alpha_1(a) = \alpha/\beta$, $\alpha_2(a) = \beta$; G acts by right translation. Let f be a K-biinvariant, compactly supported function. Define its Satake transform $f^\vee$ by $f^\vee(\pi) = \mathrm{tr}\,\pi(f)$.

Let $\pi'_\zeta = I_{H,\chi_0}(\zeta, -\zeta)$ be the representation of the group H which is normalizedly induced from the character

$$\begin{pmatrix} a & n \\ 0 & b \end{pmatrix} \longmapsto \left|\frac{a}{b}\right|^\zeta \chi_0\!\left(\frac{a}{b}\right)$$

of B'.

Let f' be a K'-biinvariant, compactly supported function on H. Its Satake transform is defined again by $f'^\vee(\pi') = \mathrm{tr}\,\pi'(f')$.

Definition. The K-biinvariant function f on G and the K'-biinvariant function f' on H are called *corresponding* if for any complex number ζ we have

$$f^{\vee}(\pi_\zeta) = f'^{\vee}(\pi'_\zeta),$$

where $\pi_\zeta = I_G(\zeta, 1/2 + \zeta)$ and $\pi'_\zeta = I_{H,\chi_0}(\zeta, -\zeta)$ are the representations of G and H defined above.

The unit elements f^0 and f'^0 of the Hecke algebras $C_c(K\backslash G/K)$ and $C_c(K'\backslash H/K')$, which are the characteristic functions of K and K' divided by their volumes, are corresponding. It is shown in [FM] that they are matching. The main result of this paper is the following extension of that result, conjectured in [FM].

Theorem. *Corresponding f and f' are matching.*

Our approach is analogous to that of [J], [M]. An alternative approach would be to directly compute the integral. The general structure of the proof is as follows. Under the action of K, S can be decomposed into K-orbits. Each orbit has a representative (see Proposition 1.7) of the form $d_r v_1$ ($r \geq 0$), where $d_r = \mathrm{diag}(\boldsymbol{\pi}^r, 1, 1, 1, \boldsymbol{\pi}^{-r})$. In the split case $v_1 = {}^t(1/2, 0, 0, 0, 1)$, in the non-split case $v_1 = {}^t(2\theta, 0, 0, 1, 1)$. The main result of Section I is Proposition 1.8, which computes the volume of $Kd_r v_1$. This section contains also some results needed in Section II.

We write $\mathcal{F}(\phi) = \phi'$ (where $\phi \in C_c(K\backslash S)$ and $\phi' \in C_c(N'\backslash H/K', \psi_{N'})$) if

$$\int_S \phi(s) T_\zeta(s) ds = \int_{A'} \phi'(a) W_\zeta(a) |a|^{-1} da.$$

Here T_ζ is the K-invariant function on S, such that $T_\zeta(d_0 v_1) = 1$ and for $r \geq 1$ (see Proposition 2.2)

$$T_\zeta(d_r v_1) = \sum_{\xi \in \{\zeta, -\zeta\}} \frac{(q^{\frac{3}{2}+\xi} - 1)(1 \mp q^{-\frac{1}{2}-\xi})}{(q^\xi - q^{-\xi})(q^{\frac{3}{2}} \mp q^{-\frac{1}{2}})} q^{-r(\frac{3}{2}-\xi)},$$

where the "$-$" sign occurs in the split case and "$+$" in the non-split case. The function W_ζ is the normalized unramified Whittaker function in the space of the representation π'_ζ (see Proposition 2.4).

We show in Proposition 2.5 that $\mathcal{F}$ is a linear bijection between the spaces $C_c(K\backslash S)$ and $C_c(N'\backslash H/K', \psi_{N'})$.

The map $f \mapsto \phi_f$ from $C_c(K\backslash G/K)$ to $C_c(K\backslash S)$, and the map $f' \mapsto \phi'_{f'}$ from $C_c(K'\backslash H/K')$ to $C_c(N'\backslash H/K', \psi_{N'})$, are used in Section II to show that the relation $f^{\vee}(\pi_\zeta) = f'^{\vee}(\pi'_\zeta)$ is equivalent to $\mathcal{F}(\phi_f) = \phi'_{f'}$.

Define ϕ_r to be the characteristic function of the orbit $Kd_r v_1$. Since $K\backslash S$ is the disjoint union of $Kd_r v_1$, $r \geq 1$, (Proposition 1.3) the set $\{\phi_r; r \geq 0\}$ is a basis of the space

$C_c(K\backslash S)$. Define $\Phi_r = \sum_{i=0}^{r} \phi_i$. Then the set $\{\Phi_r; r \geq 0\}$ is also a basis of $C_c(K\backslash S)$.

Now, we consider the group H. Since $H = N'A'K'$, any function in $C_c(N'\backslash H/K', \psi_{N'})$ is defined by its values on A'. For $r \geq 0$, define the function ϕ'_r in this space by

$$\phi'_r\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} 1, & \text{if } |\alpha| = q^{-r}, \\ 0, & \text{otherwise }. \end{cases}$$

We show in Proposition 2.4(2) that $\phi'(\mathrm{diag}(\alpha, 1)) = 0$ if $|\alpha| > 1$. Hence, the set $\{\phi'_r; r \geq 0\}$ is a basis of $C_c(N'\backslash H/K', \psi_{N'})$. Without lost of generality we assume that $\chi_0(\boldsymbol{\pi}) = -1$. Indeed, if this is not the case then we can change the basis $\phi'_r \mapsto (-1)^r \phi'_r$. Set $\phi'_r = 0$ if $r < 0$. The main result of Section II asserts that for any integer $r \geq 0$, we have $\mathcal{F}(\Phi_r) = (-1)^r q^r (\phi'_r \pm \phi'_{r-1})$, i.e.

$$\int_S \Phi_r(s) T_\zeta(s) ds = (-1)^r q^r \int_{A'} (\phi'_r(a) \pm \phi'_{r-1}(a)) W_\zeta(a) |a|^{-1} da,$$

where as usual the "+" sign occurs in the split case, and the "−" in the non-split case. Thus, if two spherical functions $f \in C_c^\infty(G)$ and $f' \in C_c^\infty(H)$ are corresponding we have $\mathcal{F}(\phi_f) = \phi'_{f'}$. Since $\mathcal{F} : C_c(K\backslash S) \to C_c(N'\backslash H/K', \psi_{N'})$ is an isomorphism of two vector spaces, to prove that corresponding functions are matching (i.e. $\Psi(\alpha, \phi_f) = (\theta, \alpha)\psi(\alpha)|\alpha|\Psi'(\alpha^{-1}, \phi'_{f'})$) it is enough to show that (for $r \geq 0$)

$$\Psi(\alpha, \Phi_r) = (\theta, \alpha)\psi(\alpha)|\alpha|\Psi'(\alpha^{-1}, (-1)^r q^r (\phi'_r \pm \phi'_{r-1})).$$

In Section III we show that (for any $r \geq 1$)

$$\int_{F^\times} \Psi(\alpha, \Phi_r)\chi(\alpha) d^\times \alpha = \int_{F^\times} (\theta, \alpha)\psi(\alpha)|\alpha|\Psi'(\alpha^{-1}, (-1)^r q^r (\phi'_r \pm \phi'_{r-1}))\chi(\alpha) d^\times \alpha,$$

where χ is any complex valued character of $F^\times$. The case $r = 0$ follows from this result and from the case of the unit element, treated in [FM]. If χ is ramified both integrals are equal to 0. The Fourier inversion formula now implies the required result for the split case.

I. The group G, subgroup C and the K-orbits of G/C.

I.1. The group G. The group $\mathbf{G} = SO(3,2)$ can also be defined as

$$\{g \in GL(5);\ Q(gv, gv) = Q(v, v),\ \det(g) = 1\},$$

where $Q(v, w) = {}^t v J w$, hence $Q(v, v) = {}^t v J v = 2v_1 v_5 + 2v_2 v_4 + v_3^2$ is a quadratic form on the 5 dimensional vector space V of columns. Let $\mathbf{P}$ be the maximal upper triangular

parabolic subgroup of $\mathbf{G}$ with abelian unipotent radical, $\mathbf{N}$. Let $\mathbf{B}$ be the upper triangular Borel subgroup of $\mathbf{G}$.

Definition. Define the matrix

$$n = n(x_1, x_2, x_3, x_4) = \begin{pmatrix} 1 & x_3' & x_2' & x_1' & z \\ 0 & 1 & -x_4 & -\frac{1}{2}x_4^2 & x_1 \\ 0 & 0 & 1 & x_4 & x_2 \\ 0 & 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

where

$$x_3' + x_3 = 0,\ x_2 + x_2' = x_3x_4,\ x_1 + x_1' = -x_2x_4 + \frac{1}{2}x_3x_4^2,\ z = -x_1x_3 - \frac{1}{2}x_2^2. \tag{2}$$

Let $\mathbf{A}_0 = \{\mathrm{diag}(\alpha, \beta, 1, \beta^{-1}, \alpha^{-1});\ \alpha, \beta \neq 0\}$ be the diagonal subgroup of $\mathbf{G}$. Let $\mathbf{N}_0 = \{n = n(x_1, x_2, x_3, x_4)\}$ be the upper triangular maximal unipotent subgroup of $\mathbf{B}_0$. Put $n(x_1, x_2, x_3) = n(x_1, x_2, x_3, 0)$. We have $\mathbf{B} = \mathbf{A}_0\mathbf{N}_0$ and $\mathbf{P} = \mathbf{NMA}$, where

$$\mathbf{M} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{pmatrix};\ m \in SO(J_3) \right\},\quad \mathbf{A} = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix};\ \alpha \neq 0 \right\},$$

and $\mathbf{N} = \{n = n(x_1, x_2, x_3)\}$. The standard Levi subgroup of $\mathbf{P}$ is the product $\mathbf{MA}$.

If $x_4 = 0$ the condition (2) reduces to $x_i' + x_i = 0$ $(i = 1, 2, 3)$ and $z = -x_1x_3 - \frac{1}{2}x_2^2$. We define $G = \mathbf{G}(F)$, $P = \mathbf{P}(F)$, $N = \mathbf{N}(F)$, $A = \mathbf{A}(F)$, $N_0 = \mathbf{N}_0(F)$ and $A_0 = \mathbf{A}_0(F)$. Define the character ψ_N on N. In the split case, let $\psi_N(n(x_1, x_2, x_3)) = \psi(x_2)$. In the non-split case, let $\psi_{N,\theta}(n(x_1, x_2, x_3)) = \psi(x_1 + 2\theta x_3)$.

Consider the split case. Put

$$\mathbf{C} = \left\{ \begin{pmatrix} A_1 & 0 & A_2 \\ 0 & 1 & 0 \\ A_3 & 0 & A_4 \end{pmatrix};\ \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in SO(J_4) \right\}.$$

Lemma 1.1. *Put $v_0 = {}^t(0, 0, 1, 0, 0)$, and $C = \mathbf{C}(F)$. Then C is the stabilizer of v_0 under the action of G on V, and the map $g \mapsto gv_0$ embeds G/C into S, where S is the sphere v in V such that $Q(v, v) = 1$.*

Proof. Clearly $C = \mathrm{Stab}_G(v_0)$. Since G is the group $SO(J)$, and the third column x of any element g of G is gv_0, x satisfies the condition $Q(x, x) = 1$. $\square$

Remark. Note that

$$S = \{{}^t(x_1, x_2, x_3, x_4, x_5) \in V;\ 2x_1x_5 + 2x_2x_4 + x_3^2 = 1\}.$$

I.2. The isomorphism between $SO(3,2)$ and $PGSp(4)$. Let G' be the group $PGSp(4,F)$ of matrices $g \in GL(4,F)$ such that $gJ'^tg = \lambda J'$, where J' is the matrix antidiag$(1,1,-1,-1)$ and $\lambda \in F^\times$. Fix $\theta \in F^\times$ which is not a square. Let $a_\theta =$ antidiag$(1,\theta)$. Let C'_θ be the centralizer of diag(a_θ, a_θ) in G'. Let N' be the unipotent radical of the Siegel parabolic subgroup P' of type $(2,2)$ of G'. Recall that

$$N' = \left\{ n = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \in G';\ A = \begin{pmatrix} x & y \\ z & x \end{pmatrix} \right\}.$$

Fix a complex valued non-trivial character ψ of F and define the character ψ_θ of N' by $\psi_\theta(n) = \psi(z - \theta y)$. The stabilizer of this character is a non-split torus (see [FM]).

Definition. Define a five dimensional space X by

$$X = \{T \in M_4(F);\ {}^t(TJ') = -TJ', \mathrm{tr}(T) = 0\}.$$

Choose the basis $\{e_1, e_2, e_3, e_4, e_5\}$ of X so that

$$T = T(x_1, x_2, x_3, x_4, x_5) = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5$$

is represented by the matrix

$$\begin{pmatrix} -x_3/2 & x_4 & x_1/2 & 0 \\ x_2/2 & x_3/2 & 0 & -x_1/2 \\ x_5 & 0 & x_3/2 & x_4 \\ 0 & -x_5 & x_2/2 & -x_3/2 \end{pmatrix}. \tag{3}$$

The inner form on this space is $(T_1, T_2) = \mathrm{tr}(T_1T_2)$, where T_1, T_2 are in X. Define an action of G' on X via g: $T \mapsto gTg^{-1}$.

Lemma 1.2. *The action g: $T \mapsto gTg^{-1}$ of $G' = PGSp(4)$ on X is well defined and establishes an isomorphism from $G' = PGSp(4)$ to $G = SO(3,2) = SO(J)$.*

Proof. To show that this action is well defined, we have to prove that ${}^t(gTg^{-1}J') = -gTg^{-1}J'$. This relation is equivalent to ${}^tJ'^tg^{-1t}T^tg = -gTg^{-1}J'$. Multiplying both sides by g^{-1} on the left and by ${}^tg^{-1}$ on the right, we obtain $g^{-1t}J'^tg^{-1t}T = -Tg^{-1}J'^tg^{-1}$. But $g^{-1}J'^tg^{-1} = \lambda J'$ implies that $g^{-1t}J'^tg^{-1} = \lambda^tJ'$. We arrive at

$$\lambda^tJ'^tT = -Tg^{-1}J'^tg^{-1} = -\lambda TJ',$$

which is true since $T \in X$.

Further if $T_1 = T_1(x_1, x_2, x_3, x_4, x_5)$ and $T_2 = T_2(y_1, y_2, y_3, y_4, y_5)$ then

$$\mathrm{tr}(T_1T_2) = x_1y_5 + x_2y_4 + x_3y_3 + x_4y_2 + x_5y_1.$$

Since $\mathrm{tr}(gT_1T_2g^{-1}) = \mathrm{tr}(T_1T_2)$, the action of G' on X defines an orthogonal group on X. The space X is isomorphic to the 5 dimensional vector space V, from Section I.1. Thus this orthogonal group is the group $G = SO(J)$. $\square$

Lemma 1.3. *Under the isomorphism of Lemma 1.1, the image of subgroup C'_θ is C_θ, the centralizer (in G) of*

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\theta & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{1}{2}\theta^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Proof. By matrix multiplication

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \theta & 0 \end{pmatrix} \begin{pmatrix} -x_3/2 & x_4 & x_1/2 & 0 \\ x_2/2 & x_3/2 & 0 & -x_1/2 \\ x_5 & 0 & x_3/2 & x_4 \\ 0 & -x_5 & x_2/2 & -x_3/2 \end{pmatrix} \begin{pmatrix} 0 & \theta^{-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta^{-1} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x_3/2 & x_2/(2\theta) & -x_1/2 & 0 \\ \theta x_4 & -x_3/2 & 0 & x_1/2 \\ -x_5 & 0 & -x_3/2 & x_2/(2\theta) \\ 0 & x_5 & \theta x_4 & x_3/2 \end{pmatrix},$$

which implies the lemma. $\square$

Note that under the isomorphism between G' and G, the unipotent subgroup N' of G' is isomorphic to the subgroup N of G via

$$\begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & z & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -y & -2x & 2z & 2zy - 2x^2 \\ 0 & 1 & 0 & 0 & -2z \\ 0 & 0 & 1 & 0 & 2x \\ 0 & 0 & 0 & 1 & y \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In particular $z - \theta y \mapsto -\frac{1}{2}(x_1 + 2\theta x_3)$ which justifies the choice of the character $\psi_{N,\theta}$ on N.

Lemma 1.4. *Put $v_0 = {}^t(0, 2\theta, 0, 1, 0)$. Then C_θ is the stabilizer of v_0 under the action of G on V, and the map $g \mapsto gv_0$ embeds G/C_θ into S, where S is the sphere v in V such that $Q(v, v) = 4\theta$.*

Proof. The image of the subgroup C'_θ in G is the subgroup C_θ, which consists of matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & -2\theta a_{12} & a_{15} \\ a_{21} & a_{22} & a_{23} & 2\theta(1-a_{22}) & a_{25} \\ a_{31} & a_{32} & a_{33} & -2\theta a_{32} & a_{35} \\ -a_{21}/(2\theta) & (1-a_{22})/(2\theta) & -a_{23}/(2\theta) & a_{22} & -a_{25}/(2\theta) \\ a_{51} & a_{52} & a_{53} & -2\theta a_{52} & a_{55} \end{pmatrix}.$$

Clearly $C_\theta = \mathrm{Stab}_G(v_0)$. Recall that $Q(v,w) = {}^t vJw$. If $\mathbf{y}_2$ is the second and $\mathbf{y}_4$ is the fourth columns of the orthogonal group G, then they satisfy $Q(\mathbf{y}_2,\mathbf{y}_2) = Q(\mathbf{y}_4,\mathbf{y}_4) = 0$ and $Q(\mathbf{y}_2,\mathbf{y}_4) = Q(\mathbf{y}_4,\mathbf{y}_2) = 1$. The element gv_0 is the sum $2\theta\mathbf{y}_2 + \mathbf{y}_4$. Hence, we have

$$Q(2\theta\mathbf{y}_2 + \mathbf{y}_4, 2\theta\mathbf{y}_2 + \mathbf{y}_4) = 4\theta^2 Q(\mathbf{y}_2,\mathbf{y}_2) + 4\theta Q(\mathbf{y}_2,\mathbf{y}_4) + Q(\mathbf{y}_4,\mathbf{y}_4) = 4\theta.$$

$\square$

Remark. The sphere S is equal to

$$S = \{{}^t(x_1,x_2,x_3,x_4,x_5) \in V;\ 2x_1x_5 + 2x_2x_4 + x_3^2 = 4\theta\}.$$

I.3. Double coset decomposition. In the split case, we define

$$v_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & \frac{1}{2} \end{pmatrix}, \quad v_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

In the non-split case ($\theta \notin (F^\times)^2$), we define

$$v_1 = \begin{pmatrix} 2\theta \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad v_0 = \begin{pmatrix} 0 \\ 2\theta \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Note that $v_1 = \gamma_0 v_0$.

Proposition 1.5. *We have the disjoint decomposition: $G = PC_\theta \cup NA\gamma_0C_\theta$. The representatives of the N-C_θ-orbits of maximal dimension, which is 9, are of the form $a\gamma_0$, $a \in A$. Futhermore, the map $g \mapsto gv_0$ of Lemmas 1.1 and 1.3 establishes an isomorphism of homogeneous spaces from G/C_θ to S.*

Proof. Consider the left action of P on S. Since the last row of any element of P is of the form $(0,0,0,0,\alpha^{-1})$, $\alpha \neq 0$, we conclude that there are at least two P-invariant subsets in S: a closed subset $\{x = {}^t(x_1,x_2,x_3,x_4,0);\ Q(x,x) = 1\}$, and an open subset $\{x = {}^t(x_1,x_2,x_3,x_4,x_5);\ Q(x,x) = 1,\ x_5 \neq 0\}$. We claim that P acts transitively on each of these two subsets.

Consider the split case. The element $v_1 = {}^t(1/2,0,0,0,1)$, which is a representative of the open subset. Acting first by an element from A and then from N, we obtain the transpose of

$$\left(\frac{\alpha}{2} + \frac{z}{\alpha}, \frac{x_1}{\alpha}, \frac{x_2}{\alpha}, \frac{x_3}{\alpha}, \frac{1}{\alpha}\right).$$

When (x_1,x_2,x_3) runs through F^3 and α over $F^\times$, this column runs through all elements ${}^t(x_1,x_2,x_3,x_4,x_5)$ of S, with $x_5 \neq 0$. Thus P acts transitively on the open P-subset of S, i.e. it is a P-orbit.

Consider the non-split case. The element $v_1 = {}^t(2\theta,0,0,1,1)$ is a representative of the open subset. Acting first by an element from A and then from N, we obtain the transpose of

$$\left(2\alpha\theta - x_1 + \frac{z}{\alpha}, \frac{x_1}{\alpha}, \frac{x_2}{\alpha}, 1 + \frac{x_3}{\alpha}, \frac{1}{\alpha}\right).$$

When (x_1,x_2,x_3) runs through F^3 and α over $F^\times$, this column runs through all elements ${}^t(x_1,x_2,x_3,x_4,x_5)$ of S, with $x_5 \neq 0$. Thus P acts transitively on the open P-invariant subset of S, making it a P-orbit.

Consider $v_0 = {}^t(0,0,1,0,0)$ in the split case and $v_0 = {}^t(0,2\theta,0,1,0)$ in the non-split case. This is a representative of the closed subset. First, acting by an element $n(0,-s,0) \in N$ (see I.1), in the split case, and $n(-s,0,0)$ in the non-split case, we can send v_0 to ${}^t(s,0,1,0,0)$ (respectively ${}^t(s,2\theta,0,1,0)$), s arbitrary. Multiplying it on the left by an element $g = \operatorname{diag}(1,m,1) \in M$, we obtain ${}^t(s,m_{21},m_{22},m_{23},0)$, where ${}^t(m_{21},m_{22},m_{32})$ is the second column of m. Thus, we obtain all elements ${}^t(x_1,x_2,x_3,x_4,0) \in S$. Note that the N-orbits in S of such elements are of dimension 1.

Consequently, the decomposition of G with respect to P and C_θ is $G = PC_\theta \cup P\gamma_0 C_\theta$. Recall that $P = NAM$. We assert that $\gamma_0^{-1} M \gamma_0 \subset C_\theta$. Indeed, from $M \subset \operatorname{Stab}(v_1)$ it follows that $\gamma_0^{-1} M \gamma_0 \subset \operatorname{Stab}(v_0)$, since $v_1 = \gamma_0 v_0$. Thus, $G = PC \cup NA\gamma_0 C$.

Since $A \subset C_\theta$, we have $PC_\theta = NMC_\theta$. We have seen that the N-orbits of NMC_θ/C_θ are of dimension 1. Hence the N-C_θ-double cosets of PC_θ are of dimension 7.

We have seen above that the map $g \mapsto gv_0$ is an onto map with kernel C_θ. Consequently $G/C_\theta \simeq S$. The proposition follows. □

I.4. The subset Kb_1K. Set $b_1 = \operatorname{diag}(\boldsymbol{\pi}^{-1},1,1,1,\boldsymbol{\pi})$, $b_2 = \operatorname{diag}(1,\boldsymbol{\pi}^{-1},1,\boldsymbol{\pi},1)$ and consider the double coset Kb_1K. Recall that $\mathbb{F}$ is the residue field of F, i.e. a finite field

of q elements, q is odd. Define $\mathbb{N}_0 = \mathbf{N_0}(\mathbb{F})$. More explicitly

$$\mathbb{N}_0 = \left\{ n(x_1, x_2, x_3, x_4) = \begin{pmatrix} 1 & x_3' & x_2' & x_1' & z \\ 0 & 1 & -x_4 & -\frac{1}{2}x_4^2 & x_1 \\ 0 & 0 & 1 & x_4 & x_2 \\ 0 & 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};\ x_1, x_2, x_3, x_4 \in \mathbb{F} \right\},$$

where z, x_4, x_3', x_3, x_2', x_2, x_1', x_1 satisfy the relations (2). Define in $\mathbb{N}_0$ the subgroups (the order of N_i is q^{4-i}):

$$\begin{aligned} N_1 &= \{n \in \mathbb{N}_0;\ n = n(x_1, x_2, x_3, 0)\}, \\ N_2 &= \{n \in \mathbb{N}_0;\ n = n(x_1, 0, 0, x_4)\}, \\ N_3 &= \{n \in \mathbb{N}_0;\ n = n(0, 0, x_3, 0)\}. \end{aligned}$$

We regard N_i and $\mathbb{N}_0$ as subsets of $\mathbf{N}_0(\mathbb{R})$ on choosing representatives in R of the elements of $\mathbb{F} = R/\boldsymbol{\pi}$.

The following Proposition is used in the proof of Proposition 1.8.

Proposition 1.6. *We have the disjoint decomposition*

$$Kb_1K = Kb_1N_1 \cup Kb_2N_2 \cup Kb_2^{-1}N_3 \cup Kb_1^{-1}. \tag{4}$$

Proof. The Weyl group $W = \{1, (15), (24), (12)(45), (14)(25), (15)(24), (1452), (1254)\}$ embeds in K, so we have

$$Kb_1K = Kb_1^{-1}K = Kb_2K = Kb_2^{-1}K.$$

Let W_1 be the subset $\{1, (15), (12)(45), (14)(25)\}$ of W, and $P_1 = K \cap b_1Kb_1^{-1}$ a parahoric (see, e.g., [T]) subgroup of K. Using the Iwahori decomposition:

$$K = P_1W_1P_1 = \cup_{w \in W_1} P_1wP_1,$$

we have

$$Kb_1^{-1}K = \cup_{w \in W_1} Kb_1^{-1}P_1wP_1.$$

Since $b_1^{-1}P_1b_1 \in K$, this is equal to

$$\cup_{w \in W_1} Kb_1^{-1}wP_1.$$

Since $W_1 \subset K$, and $\{w^{-1}b_1^{-1}w; w \in W_1\}$ is the set $\{b_1^{-1}, b_1, b_2^{-1}, b_2\}$, we obtain

$$Kb_1^{-1}P_1 \cup Kb_1P_1 \cup Kb_2^{-1}P_1 \cup Kb_2P_1.$$

In our analysis below, we use the decomposition $P_1 = N' M_K A_K N_K$, where N' is the subgroup of ${}^tN$ with underdiagonal entries from $\boldsymbol{\pi} R$, $A_K = A \cap K$, $N_K = N \cap K$ and $M_K = M \cap K$ is the maximal compact of $M = \{\mathrm{diag}(1, m, 1); m \in SO(J_3)\}$.

To describe the four double cosets, we introduce:

Case of $Kb_1^{-1}P_1$. Since $b_1^{-1}P_1 b_1 \subset K$, we have $Kb_1^{-1}P_1 = Kb_1^{-1}$.

Case of Kb_1P_1. Since $b_1 N' b_1^{-1} \subset K$ and $M_K A_K$ commutes with b_1, we have $Kb_1P_1 = Kb_1N_K$.

The set of elements $n \in N_K$ with entries above the diagonal from $\boldsymbol{\pi} R$ is a normal subgroup of N_K of elements satisfying $b_1 n b_1^{-1} \in K$. Since the quotient of N_K by this subgroup is N_1, we have $Kb_1N_K = Kb_1N_1$.

Case of $Kb_2^{-1}P_1$. Since $b_2^{-1}N'b_2 \subset K$, we have $Kb_2^{-1}P_1 = Kb_2^{-1}M_KN_K$.

Let W_2 be the subgroup of two elements $\{1, (13)\}$, where (13) is represented by the matrix $w_2 = \mathrm{antidiag}(1, -1, 1)$, and put $P_2 = M_K \cap b_2 M_K b_2^{-1}$. Using the Iwahori decomposition $M_K = P_2 W_2 P_2$, our set is $Kb_2^{-1}P_2W_2P_2N_K$. Since $b_2^{-1}P_2b_2 \in K$, this is $Kb_2^{-1}W_2P_2N_K$. But $W_2 = \{1, w_2\}$ and $w_2 b_2^{-1} w_2 = b_2$, so this double coset is

$$Kb_2^{-1}P_2N_K \cup Kb_2^{-1}w_2P_2N_K = Kb_2^{-1}N_K \cup Kb_2P_2N_K.$$

Subcase of $Kb_2^{-1}N_K$. If $n \in N_K$ is such that $b_2^{-1}nb_2 \in K$, then $n = n(x_1, x_2, x_3, x_4)$, where $x_1, x_2, x_4 \in R$ and $x_3 \in \boldsymbol{\pi} R$. The set of such elements is a normal subgroup of N_K. Since the quotient of N_K by this subgroup is N_3, we obtain

$$Kb_2^{-1}N_K = Kb_2^{-1}N_3.$$

Subcase of $Kb_2P_2N_K$. To simplify the notations, write $m \in GL(3)$ for $\mathrm{diag}(1, m, 1) \in GL(5)$. Then $P_2 = U'A_2U$, where we put $A_2 = \{\mathrm{diag}(\alpha, 1, \alpha^{-1}); \alpha \in F^\times, |\alpha| = 1\}$,

$$U' = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ -\frac{1}{2}x^2 & x & 1 \end{pmatrix};\ x \in \boldsymbol{\pi} R \right\},\ U = \left\{ \begin{pmatrix} 1 & -y & -\frac{1}{2}y^2 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix};\ y \in R \right\}.$$

Since $b_2U'b_2^{-1} \subset K$ and A_2 commutes with b_2, the coset $Kb_2P_2N_K$ is equal to

$$Kb_2U'A_2UN_K = Kb_2UN_K.$$

Note that $UN_K = N_0 \cap K$. Finally, any $n \in N_0$ such that $b_2nb_2^{-1} \in K$, is of the form $n = n(x_1, x_2, x_3, x_4)$, where $x_2, x_3 \in R$ and $x_1, x_4 \in \boldsymbol{\pi} R$. The set of such elements is a subgroup of $\mathbf{N}_0(R)$, and N_2 is the set of representatives of its left cosets in $\mathbf{N}_0(R)$. Thus $Kb_2P_2N_K = Kb_2N_2$.

Case of Kb_2P_1. Decomposing P_1 and using $b_2N'b_2^{-1} \subset K$ and $b_2A_Kb_2^{-1} = A_K \subset K$, we obtain

$$Kb_2P_1 = Kb_2N'A_KM_KN_K = Kb_2M_KN_K.$$

Applying the Iwahori decomposition to M_K, ($W_2 = \{1, w_2\}$), this is

$$Kb_2P_2W_2P_2N_K = Kb_2P_2N_K \cup Kb_2P_2w_2P_2N_K.$$

The double coset $Kb_2P_2N_K$ has already been considered. We are left with the double coset $Kb_2P_2w_2P_2N_K$.

Since $P_2 = U'A_2U$ and $w_2U'A_2w_2 \subset P_2$, we have

$$P_2w_2P_2 = P_2w_2U'A_2U = P_2w_2U.$$

Futhermore, since $b_2U'A_2b_2^{-1} \subset K$, the double coset $Kb_2P_2w_2P_2N_K$ is equal to

$$Kb_2P_2w_2UN_K = Kb_2U'A_2Uw_2UN_K = Kb_2Uw_2UN_K.$$

Since $w_2^{-1}b_2w_2 = b_2^{-1}$ this is equal to

$$Kb_2w_2w_2^{-1}Uw_2UN_K = Kb_2^{-1}U_1UN_K, \quad \text{where } U_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ -y & 1 & 0 \\ -\frac{1}{2}y^2 & y & 1 \end{pmatrix}; \ y \in \mathbb{F} \right\}.$$

According to the Iwasawa decomposition, any element $n \in U_1$ can be written as $n = u_1sw_2u_2$, where $u_i \in U$, and s, the diagonal matrix, can be commuted across b_2^{-1}. Since $w_2^{-1}b_2^{-1}w_2 = b_2$, we have

$$Kb_2^{-1}U_1UN_K = Kb_2UN_K,$$

obtaining again a coset which has already been considered. □

I.5. The K-orbits of S. Under the action of K, the sphere $S = \{x \in V; Q(x,x) = 1\}$ can be decomposed as a union of open and closed K-orbits. The following Proposition is the special case of a more general statement (see [MS, Prop. 3.9]).

Proposition 1.7. *Set $d_r = \operatorname{diag}(\boldsymbol{\pi}^r, 1, 1, 1, \boldsymbol{\pi}^{-r})$. Each K-orbit of S is of the form Kd_rv_1, $r \geq 0$. The element ${}^t(x_1, x_2, x_3, x_4, x_5) \in S$ belongs to the K-orbit of d_rv_1 if and only if*

$$\|x\| = \max\{|x_1|, |x_2|, |x_3|, |x_4|, |x_5|\} \text{ is equal to } q^r.$$

Proof. Consider the set Kd_rv_1. If $\mathbf{k}_i$ is the ith column of $k \in K$, then kd_rv_1 is equal to $\frac{1}{2}\mathbf{k}_1\boldsymbol{\pi}^r + \mathbf{k}_5\boldsymbol{\pi}^{-r}$. When k ranges over all elements of K, this sum ranges over all elements ${}^t(x_1, x_2, x_3, x_4, x_5) \in S$ such that $\max\{|x_1|, |x_2|, |x_3|, |x_4|, |x_5|\} = q^r$, since the max absolute value of the entries of k_i is 1. □

Let ϕ_r be the characteristic function of the K-orbit of d_rv_1 in S. Normalize the additive measure dx on F and the multiplicative measure $d^\times x$ on $F^\times$ ($d^\times x = (1-1/q)^{-1}|x|^{-1}dx$), so that

$$\int_R dx = 1, \qquad \text{and} \qquad \int_{|x|=1} d^\times x = (1-1/q)^{-1}\int_{|x|=1} \frac{dx}{|x|} = 1.$$

Normalize the measure on K so that its volume is 1. We need the following result.

Proposition 1.8. *The volume Λ_r of the K-orbit of $d_r v_1$ in S is given by*

$$\Lambda_r = q^{3r}(1 \mp q^{-2})\Lambda_0, \quad \text{if} \ \ r \geq 1, \tag{5}$$

where the "$-$" sign is in the split case and "$+$" in the non-split case.

Proof. Suppose $r \geq 1$. Let f_1 be the characteristic function of Kb_1K in G. Since the measure ds on S is invariant under the action of G, we have

$$\int_G \int_S \phi_r(gs) ds f_1(g) dg = \int_G f_1(g) dg \int_S \phi_r(s) ds. \tag{6}$$

Using Proposition 1.6, and that $\#N_i = q^{4-i}$, the right hand side of (6) is equal to

$$(q^3 + q^2 + q + 1)\Lambda_r. \tag{7}$$

In the left hand side of (6), we change the order of integration. It is

$$\int_S I_r(s) ds, \quad \text{where} \ \ I_r(s) = \int_G \phi_r(gs) f_1(g) dg.$$

Note that $I_r(ks) = I_r(s)$ for any $k \in K$. Thus it is constant on the K orbits in S. In particular, if s is in the K-orbit of $d_i v_1$, we have $I_r(s) = I_r(d_i v_1)$. We obtain

$$\int_S I_r(s) ds = \sum_{i=0}^{\infty} \Lambda_i I_r(d_i v_1). \tag{8}$$

Using Proposition 1.6, the value of $I_r(d_i v_1)$ is

$$\int_G \phi_r(g d_i v_1) f_1(g) dg = \sum_{n \in N_1} \int_K \phi_r(k b_1 n d_i v_1) dk + \sum_{n \in N_2} \int_K \phi_r(k b_2 n d_i v_1) dk$$

$$+ \sum_{n \in N_3} \int_K \phi_r(k b_2^{-1} n d_i v_1) dk + \int_K \phi_r(k b_1^{-1} d_i v_1) dk.$$

Since ϕ_r is left K-invariant, the expression above is equal to

$$\sum_{n \in N_1} \phi_r(b_1 n d_i v_1) + \sum_{n \in N_2} \phi_r(b_2 n d_i v_1) + \sum_{n \in N_3} \phi_r(b_2^{-1} n d_i v_1) + \phi_r(b_1^{-1} d_i v_1). \tag{9}$$

We consider the contribution to (8) from each sum of (9). In each case we distinguish between $r \geq 1$ and $r = 0$. In some cases we consider the case $r = 1$ separately.

Case 1. Consider the contribution to (8) from the first sum in (9). In this case $n \in N_1$ and the element $b_1 n d_i v_1$, in the split case, is equal to

$$\begin{pmatrix} \boldsymbol{\pi}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{\pi} \end{pmatrix} \begin{pmatrix} 1 & -x_3 & -x_2 & -x_1 & z \\ 0 & 1 & 0 & 0 & x_1 \\ 0 & 0 & 1 & 0 & x_2 \\ 0 & 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\boldsymbol{\pi}^i \\ 0 \\ 0 \\ 0 \\ \boldsymbol{\pi}^{-i} \end{pmatrix}.$$

In the non-split case the last vector column is

$$^t(2\theta\boldsymbol{\pi}^i, 0, 0, 1, \boldsymbol{\pi}^{-i}).$$

These elements are equal to (split/non-split cases respectively)

$$^t(\boldsymbol{\pi}^{i-1}/2 + z\boldsymbol{\pi}^{-(i+1)}, x_1\boldsymbol{\pi}^{-i}, x_2\boldsymbol{\pi}^{-i}, x_3\boldsymbol{\pi}^{-i}, \boldsymbol{\pi}^{1-i}), \tag{10}$$

$$^t(2\theta\boldsymbol{\pi}^{i-1} - x_1\boldsymbol{\pi}^{-1} + z\boldsymbol{\pi}^{-(i+1)}, x_1\boldsymbol{\pi}^{-i}, x_2\boldsymbol{\pi}^{-i}, 1 + x_3\boldsymbol{\pi}^{-i}, \boldsymbol{\pi}^{1-i}). \tag{11}$$

Depending on n, these elements belong to different K-orbits of S.

Let $r \geq 2$. Put $n(x_1, x_2, x_3) = n(x_1, x_2, x_3, 0)$. We decompose the group N_1 into a disjoint union as follows.

(i) Let $n = n(0,0,0)$ be the identity matrix. Then $b_1 n d_i v_1 = b_1 d_i v_1 = d_{i-1} v_1$. Hence $\phi_r(b_1 n d_i v_1) = 0$ unless $i - 1 = r$ (or $i = r+1$). The contribution to (8) of this case is Λ_{r+1}.

(ii) Let $n = n(x_1, x_2, x_3)$ be a non-identity matrix with $z = 0$. We claim that there are $(q^2 - 1)$ such matrices. Indeed $z = 0$ implies $2x_1x_3 + x_2^2 = 0$. The latter equation has $q(q-1)$ solutions with $x_3 \neq 0$ (x_2 arbitrary) and $q - 1$ solutions with $x_2 = x_3 = 0$, $x_1 \neq 0$. Since $z = 0$, but n is non-identity, the element $b_1 n d_i v_1$ belongs to the K-orbit of $d_i v_1$. Hence $\phi_r(b_1 n d_i v_1) = 0$ unless $i = r$. The contribution to (8) of this case is $(q^2 - 1)\Lambda_r$.

(iii) The remaining matrices $n = n(x_1, x_2, x_3)$ have $z \neq 0$. Since the order of N_1 is q^3, there are $q^3 - q^2$ such matrices. The element $b_1 n d_i v_1$ is in the K-orbit of $d_{i+1} v_1$ (since $z \neq 0$). The function $\phi_r(b_1 n d_i v_1)$ is 0 unless $i+1 = r$ (or $i = r-1$). Thus, the contribution to (8) of this case is $(q^3 - q^2)\Lambda_{r-1}$. The contribution from the cases *(i)*, *(ii)* and *(iii)* is (when $r \geq 2$)

$$(q^3 - q^2)\Lambda_{r-1} + (q^2 - 1)\Lambda_r + \Lambda_{r+1}, \tag{12}$$

Let $r = 1$. The only contributions to (8) occur when $i = 0, 1, 2$. If $i = 2$, the element $b_1 n d_2 v_1$ (see (10)) is in the K-orbit of $d_1 v_1$ precisely when n is the identity matrix. The contribution to (8) is Λ_2. If $i = 1$, the element $b_1 n d_1 v_1$ is in the K-orbit of $d_1 v_1$ when $z = 0$ and n is a non-identity matrix. As we have seen above, the equation $z = 0$ has $q^2 - 1$ solutions. The contribution is $(q^2 - 1)\Lambda_1$. If $i = 0$ the element $b_1 n v_1$ is

$$^t((1/2 + z)\boldsymbol{\pi}^{-1}, x_1, x_2, x_3, \boldsymbol{\pi}),\ ^t(\boldsymbol{\pi}^{-1}(2\theta - x_1 + z), x_1, x_2, x_3, \theta\boldsymbol{\pi}) \tag{13}$$

(split/non-split cases respectively) is in the K-orbit of d_1v_1 when $z+\frac{1}{2}\neq 0$ or $2\theta-x_1+z\neq 0$. The equation $z+\frac{1}{2}=0$ is equivalent to $2x_1x_3+x_2^2=1$, which has q^2+q solutions. Indeed, there are $q(q-1)$ solutions with $x_3\neq 0$, x_2 arbitrary, and $2q$ solutions with $x_3=0$, $x_2=\pm 1$, x_1 arbitrary.

The equation $2\theta-x_1+z=0$ (where $z=-x_1x_3-x_2^2/2$) is equivalent to $2x_1(1+x_3)+x_2^2=4\theta$, which has q^2-q solutions. Indeed, there are $q(q-1)$ solutions with $x_1\neq 0$, x_2 arbitrary, and no solutions with $x_1=0$, since 4θ is a non-square.

Hence, this case contributes $(q^3-q^2-q)\Lambda_0$. So, when $r=1$, we have

$$(q^3-q^2\mp q)\Lambda_0+(q^2-1)\Lambda_1+\Lambda_2, \tag{14}$$

where the "$-$" is in the split case and "$+$" in the non-split case.

Let $r=0$. We distinguish between two cases:

(i) Let n be the identity matrix, i.e. $x_1=x_2=x_3=0$, and $z=0$. The element $b_1d_iv_1$ (see (10)) is in the K-orbit of $d_{i-1}v_1$. We have $\phi_0(d_{i-1}v_1)=0$ unless $i=1$. This contributes (in both split and non-split cases) Λ_1 to (8).

(ii) Let n be a non-identity matrix. Since x_1, x_2, x_3 are not all zeroes, (10) is the K-orbit of $d_{-i}v_1$. Hence $\phi_0(d_{-i}v_1)=0$ unless $i=0$. Thus only $i=0$ contributes. This contribution occurs when $b_1nv_1\in Kv_1$. This happens (see (13)) when $1/2+z=0$, in the split case, or $2\theta-x_1+z=0$ in the non-split case. The first equation has q^2+q solutions, the second one q^2-q. Their contribution to (8) is $q(q\pm 1)\Lambda_0$.

Thus Case 1, with $r=0$, contributes to (8) the quantity $(q^2\pm q)\Lambda_0+\Lambda_1$.

Case 2. Consider the contribution to (8) from the second sum of (9). For $n\in N_2$, the element $b_2nd_iv_1$, in the split case is equal to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & \boldsymbol{\pi}^{-1} & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & \boldsymbol{\pi} & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -x_1 & 0\\ 0 & 1 & -x_4 & -\frac{1}{2}x_4^2 & x_1\\ 0 & 0 & 1 & x_4 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\boldsymbol{\pi}^i\\ 0\\ 0\\ 0\\ \boldsymbol{\pi}^{-i} \end{pmatrix}=\begin{pmatrix} \frac{1}{2}\boldsymbol{\pi}^i\\ x_1\boldsymbol{\pi}^{-(i+1)}\\ 0\\ 0\\ \boldsymbol{\pi}^{-i} \end{pmatrix},$$

and in the non-split case, replacing $(\boldsymbol{\pi}^i/2,0,0,0,\boldsymbol{\pi}^{-i})$ with $(2\theta\boldsymbol{\pi}^i,0,0,1,\boldsymbol{\pi}^{-i})$, we obtain

$$\left(2\theta\boldsymbol{\pi}^i-x_1,\boldsymbol{\pi}^{-1}(x_1\boldsymbol{\pi}^{-i}-\frac{1}{2}x_4^2),x_4,\boldsymbol{\pi},\boldsymbol{\pi}^{-i}\right).$$

For $r\geq 1$ in the split case and $r\geq 2$ in the non-split case, we have

(i) If $x_1=0$, the element $b_2nd_iv_1$ belongs to the K-orbit of d_iv_1. Since x_4 can be arbitrary, the contribution to (8) is $q\Lambda_r$.

(ii) If $x_1\neq 0$ (there are q^2-q such matrices), the element $b_2nd_iv_1$ belongs to the K-orbit of $d_{i+1}v_1$. We have $\phi_r(d_{i+1})=0$ unless $i+1=r$ (or $i=r-1$). Their contribution is $(q^2-q)\Lambda_{r-1}$.

Consider the non-split case with $r=1$. The contribution occurs when $i=0,1$. If $i=1$, we have that $x_1=0$ and x_4 can be arbitrary. If $i=0$, we have a contribution when

$x_1 - \frac{1}{2}x_4^2 \neq 0$, which has $q^2 - q$ solutions. The resulting contribution is the same as in case $r \geq 2$.

Thus, the contribution from this case (for $r \geq 1$) to (8) is (same in both split/non-split cases)

$$(q^2 - q)\Lambda_{r-1} + q\Lambda_r. \tag{15}$$

Now let $r = 0$. First, consider the split case. If $x_1 \neq 0$, the element $b_2 n d_i v_1$ lies in the K-orbit of $d_{i+1}v_1$. Hence there are no positive i for which $b_2 n d_i v_1$ belongs to the K-orbit of $v_1 = d_0 v_1$. If $x_1 = 0$ then $b_2 n d_i v_1$ is in the K-orbit of $d_i v_1$. We have $\phi_0(d_i v_1) = 0$ unless $i = 0$. Since x_1 is arbitrary, this case contributes $q\Lambda_0$ to (8). In the non-split case, the only contribution occurs when $i = 0$ and $x_1 - \frac{1}{2}x_4^2 = 0$. This case contributes $q\Lambda_0$ to (8).

Case 3. Consider the contribution to (8) from the third sum of (9). For $n \in N_3$, the element $b_2^{-1} n d_i v_1$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{\pi} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{\pi}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x_3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\boldsymbol{\pi}^i \\ 0 \\ 0 \\ 0 \\ \boldsymbol{\pi}^{-i} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\boldsymbol{\pi}^i \\ 0 \\ 0 \\ x_3\boldsymbol{\pi}^{-(i+1)} \\ \boldsymbol{\pi}^{-i} \end{pmatrix}$$

in the split case, and in the non-split case it is

$$\left(2\theta\boldsymbol{\pi}^i, 0, 0, \boldsymbol{\pi}^{-1}(1 + x_3\boldsymbol{\pi}^{-i}), \boldsymbol{\pi}^{-i}\right).$$

Consider $r \geq 1$. If $x_3 = 0$ the element $b_2 n d_i v_1$ lies in the K-orbit of $d_i v_1$; otherwise it is in the K-orbit of $d_{i+1}v_1$. Thus, the only contribution to (8) occurs when $i = r$ or $i = r - 1$. Since N_3 has q elements, this contribution (for both split and non-split cases) is

$$(q - 1)\Lambda_{r-1} + \Lambda_r. \tag{16}$$

If $r = 0$, the only contribution, Λ_0, to (8), occurs when $i = 0$ and $x_3 = 0$ or $x_3 + 1 = 0$ in the split and non-split cases respectively.

Case 4. The element $b_1^{-1} d_i v_1$ is in the K-orbit of $d_{i+1}v_1$. When $r \geq 1$, it contributes the term Λ_{r-1} to (8). There is no contribution when $r = 0$.

In all expressions below, the upper sign corresponds to the split case and the lower to a non-split case.

Summing up, when $r \geq 2$, the sum (8) is equal to the sum of (12), (15), (16) and Λ_{r-1}:

$$q^3\Lambda_{r-1} + (q^2 + q)\Lambda_r + \Lambda_{r+1}.$$

In all expressions below, the upper sign corresponds to the split case and the lower to a non-split case. When $r = 0$, the sum (8) is equal to

$$(q^2 \pm q)\Lambda_0 + \Lambda_1 + q\Lambda_0 + \Lambda_0 = (q^2 + 2q + 1)\Lambda_0 + \Lambda_1.$$

When $r = 1$, using (14) instead of (12), this sum is

$$(q^3 \mp q)\Lambda_0 + (q^2 + q)\Lambda_1 + \Lambda_2.$$

Hence, we obtained the left hand sides of the equation (6), for $r \geq 2$, $r = 0$ and $r = 1$. Using (7), when $r = 0, 1$, we have

$$\begin{aligned} (q^3 \mp q)\Lambda_0 + (q^2 + q)\Lambda_1 + \Lambda_2 &= (q^3 + q^2 + q + 1)\Lambda_1, \\ (q^2 + q \pm q + 1)\Lambda_0 + \Lambda_1 &= (q^3 + q^2 + q + 1)\Lambda_0. \end{aligned}$$

The equations imply that

$$\Lambda_1 = (q^3 \mp q)\Lambda_0 = q^3(1 \mp q^{-2})\Lambda_0, \text{ and } \Lambda_2 = q^6(1 \mp q^{-2})\Lambda_0.$$

For $r \geq 2$, we have

$$q^3\Lambda_{r-1} + (q^2 + q)\Lambda_r + \Lambda_{r+1} = (q^3 + q^2 + q + 1)\Lambda_r,$$

namely $\Lambda_{r+1} - \Lambda_r = q^3(\Lambda_r - \Lambda_{r-1})$, or

$$\Lambda_{r+1} = (1 + q^3)\Lambda_r - q^3\Lambda_{r-1}.$$

The proposition follows by induction. $\square$

II. Correspondence of spherical functions.

II.1. Satake transform on the Hecke algebra of G. Let $\pi = I_G(\zeta, \zeta')$ be the unramified representation of G normalizedly induced from the character $|\alpha_1|^{\zeta}|\alpha_2|^{\zeta'}$ of B. Define the Satake transform $f^\vee$ of a spherical function f on G (i.e. $f \in C_c^\infty(G)$ and it is K-biinvariant), by $f^\vee(\pi) = \text{tr } \pi(f)$. Then

$$\pi(f)\phi_0 = f^\vee(\pi)\phi_0,$$

where ϕ_0 is the unique – up to a scalar multiple – K-fixed vector in the space of $I_G(\zeta, \zeta')$. We fix ϕ_0 by $\phi_0(1) = 1$.

Set $\pi_\zeta = I_G(\zeta, 1/2 + \zeta)$. We shall show in Proposition 2.2 below that there exists a unique K-invariant function T_ζ on the sphere S, such that $T_\zeta(1) \neq 0$ and

$$\int_G f(g)T_\zeta(gs)dg = f^\vee(\pi_\zeta)T_\zeta(s). \tag{17}$$

Applying (17) with $s = 1$, the Satake transform of any spherical function f on G is given on the π_ζ by

$$f^\vee(\pi_\zeta) = \int_S \phi_f(s)T_\zeta(s)T_\zeta(1)^{-1}ds. \tag{18}$$

Since T_ζ is K-invariant, it will be defined by its values on the K-orbits of S, which are of the form $Kd_r v_1$, where $d_r = \mathrm{diag}(\boldsymbol{\pi}^r, 1, 1, 1, \boldsymbol{\pi}^{-r})$ and $r \geq 0$. Set $T_{r,\zeta} = T_\zeta(d_r v_1)$, $r \geq 0$. These numbers are computed in Proposition 2.2.

Put $\Phi_r = \sum_{i=0}^r \phi_i$, where ϕ_i is the characteristic function of the K-orbit of $d_i v_1$. Then, the function Φ_r is the characteristic function of a subset of S defined by

$$\{{}^t(x_1, x_2, x_3, x_4, x_5) \in S;\ |x_i| \leq q^r,\ 1 \leq i \leq 5\}.$$

The main goal of this subsection is to compute the integral $\int_S \Phi_r(s) T_\zeta(s) ds$, $r \geq 1$.

To compute the numbers $T_{r,\zeta}$, we need the following result. Recall that f_1 is the characteristic function of the double coset Kb_1K, where b_1 is $\mathrm{diag}(\boldsymbol{\pi}^{-1}, 1, 1, 1, \boldsymbol{\pi})$.

Proposition 2.1. *The Satake transform $f_1^\vee(\pi)$ at $\pi = I_G(\zeta, \zeta')$ of f_1 is*

$$f_1^\vee(\pi) = q^{3/2}(q^\zeta + q^{\zeta'-\zeta} + q^{\zeta-\zeta'} + q^{-\zeta}).$$

Proof. We follow [FM], Section F. Any $\phi \in I_G(\zeta, \zeta')$ satisfies

$$\phi(nak) = \delta_B(a)^{1/2} |\alpha_1(a)|^\zeta |\alpha_2(a)|^{\zeta'} \phi(k), \tag{19}$$

where $a = \mathrm{diag}(\alpha, \beta, 1, \beta^{-1}, \alpha^{-1})$, and α_1 and α_2 are the simple roots of $G = SO(5)$, defined by $\alpha_1(a) = \alpha/\beta$ and $\alpha_2(a) = \beta$. We have

$$(\pi(f)\phi)(h) = \int_G f(g)\phi(hg) dg.$$

Using the measure decomposition $dg = \delta_B(a)^{-1} dn da dk$ and (19), this is equal to

$$\int_{N_0} \int_{A_0} \int_K f(h^{-1}nak) \delta_B(a)^{-1/2} |\alpha_1(a)|^\zeta |\alpha_2(a)|^{\zeta'} \phi(k) dn da dk.$$

Put

$$F_f(a) = \delta_B(a)^{-1/2} \int_K \int_{N_0} f(k^{-1}ank) dn dk.$$

Then

$$\mathrm{tr}\, I(\zeta, \zeta'; f) = \int_{A_0} F_f(a) |\alpha\beta^{-1}|^\zeta |\beta|^{\zeta'} da.$$

When f is K-biinvariant,

$$F_f(a) = \delta_B(a)^{-1/2} \int_{N_0} f(an) dn.$$

According to Proposition 1.6, the double coset Kb_1K is the disjoint union

$$Kb_1N_1 \cup Kb_2N_2 \cup Kb_2^{-1}N_3 \cup Kb_1^{-1}.$$

It follows that the integral $F_{f_1}(a)$ vanishes unless a is in the K-double cosets of b_1, b_2, b_2^{-1} or b_1^{-1}. Further, $\delta_B(b_1) = q^3$, $\delta_B(b_2) = q$, $\delta_B(b_2^{-1}) = q^{-1}$ and $\delta_B(b_1^{-1}) = q^{-3}$. Hence

$$F_{f_1}(b_1) = q^{-3/2}q^3 = q^{3/2},\ F_{f_1}(b_2) = q^{-1/2}q^2 = q^{3/2},$$

$$F_{f_1}(b_2^{-1}) = q^{1/2}q = q^{3/2},\ F_{f_1}(b_1^{-1}) = q^{3/2}1 = q^{3/2}.$$

Evaluating the characters α_1, α_2 at b_1, b_2, b_2^{-1} and b_1^{-1}, we obtain

$$\int_{A_0} F_{f_1}(a)|\alpha\beta^{-1}|^{\zeta}|\beta|^{\zeta'}da = q^{3/2}(q^{\zeta} + q^{\zeta'-\zeta} + q^{\zeta-\zeta'} + q^{-\zeta}).$$

Since $f_1^{\vee}(\pi) =$ tr $I_G(\zeta, \zeta'; f_1)$, the proposition follows. $\square$

We use Proposition 2.1 to prove the following:

Proposition 2.2. *The equation (17) has a unique solution satisfying $T_{0,\zeta} = 1$, and (for $r \geq 1$), we have*

$$T_{r,\zeta} = \sum_{\xi\in\{\zeta,-\zeta\}} \frac{(q^{\frac{3}{2}+\xi} - 1)(1 \mp q^{-\frac{1}{2}-\xi})}{(q^{\xi} - q^{-\xi})(q^{\frac{3}{2}} \mp q^{-\frac{1}{2}})} q^{-r(\frac{3}{2}-\xi)},$$

where the "+" sign occurs in the split case and "−" in the non-split case.

Proof. By Proposition 2.1, the equation (17), with $s = d_r v_0$ and $f = f_1$, becomes

$$\int_G f_1(g)T_{\zeta}(gd_r v_0)dg = q^{3/2}(q^{\zeta} + q^{1/2} + q^{-1/2} + q^{-\zeta})T_{r,\zeta}. \tag{20}$$

Since f_1 is the characteristic function of the double coset $Kb_1K = Kb_1N_1 \cup Kb_2N_2 \cup Kb_2^{-1}N_3 \cup Kb_1^{-1}$, and the function T_{ζ} is invariant under K, whose volume is 1, the left hand side of (20) equals

$$\sum_{n\in N_1} T_{\zeta}(b_1 n d_r v_1) + \sum_{n\in N_2} T_{\zeta}(b_2 n d_r v_1) + \sum_{n\in N_3} T_{\zeta}(b_2^{-1} n d_r v_1) + T_{\zeta}(b_1^{-1} d_r v_1). \tag{21}$$

We will compute each of the sums.

To simplify the notations (in the proof of this proposition), we write T for T_{ζ} and T_r for $T_{r,\zeta}$. For more details see Proposition 1.8.

Case 1. Consider the first term of (21). Put $n(x_1, x_2, x_3) = n(x_1, x_2, x_3, 0)$.
Let $r \geq 1$. In the split case, we have

$$b_1 n(x_1, x_2, x_3)d_r v_1 = {}^t(\boldsymbol{\pi}^{r-1}/2 + z\boldsymbol{\pi}^{-(r+1)}, x_1\boldsymbol{\pi}^{-r}, x_2\boldsymbol{\pi}^{-r}, x_3\boldsymbol{\pi}^{-r}, \boldsymbol{\pi}^{1-r}).$$

In the non-split case, we have

$$b_1 n(x_1, x_2, x_3)d_r v_1 = {}^t(2\theta\boldsymbol{\pi}^{r-1} - x_1\boldsymbol{\pi}^{-1} + z\boldsymbol{\pi}^{-(r+1)}, x_1\boldsymbol{\pi}^{-r}, x_2\boldsymbol{\pi}^{-r}, 1 + x_3\boldsymbol{\pi}^{-r}, \boldsymbol{\pi}^{1-r}).$$

Consider the following cases:

(i) Let $n = n(0,0,0)$ be the identity matrix. Then $b_1 n d_r v_1 = b_1 d_r v_1 = d_{r-1} v_1$ and $T(b_1 n d_r v_1) = T(d_{r-1} v_1) = T_{r-1}$

(ii) Let $n = n(x_1, x_2, x_3)$ be a non-identity matrix with $z = 0$. In Proposition 1.8, we showed that there are $(q^2 - 1)$ such matrices. The element $b_1 n d_r v_1$ belongs to the K-orbit of $d_r v_1$. Hence, $T(b_1 n d_r v_1) = T(d_r v_1) = T_r$.

(iii) The remaining $q^3 - q^2$ matrices $n = n(x_1, x_2, x_3)$ have $z \neq 0$. The element $b_1 n d_r v_1$ is in the K-orbit of $d_{r+1} v_1$. Thus, $T(b_1 n d_r v_1) = T(d_{r+1} v_1) = T_{r+1}$.

We conclude that for $r \geq 1$, we have

$$\sum_{n \in N_1} T(b_1 n d_r v_1) = (q^3 - q^2) T_{r+1} + (q^2 - 1) T_r + T_{r-1}. \tag{22}$$

Let $r = 0$ ($d_0 = 1$). The element $b_1 n(x_1, x_2, x_3) v_1$ is equal to

$${}^t(\boldsymbol{\pi}^{-1}(1/2 + z), x_1, x_2, x_3, \boldsymbol{\pi}), \text{ or } {}^t(\boldsymbol{\pi}^{-1}(2\theta - x_1 + z), x_1, x_2, 1 + x_3, \boldsymbol{\pi}).$$

in the split/non-split cases respectively. These elements are in the K-orbit of $d_1 v_1$ if $\frac{1}{2} + z \neq 0$ or $2\theta - x_1 + z \neq 0$. Otherwise, they are in the K orbit of v_1.

The equation $\frac{1}{2} + z = 0$ has $q^2 + q$ solutions, and $2\theta - x_1 + z = 0$ has $q^2 - q$ solutions. Thus

$$\sum_{n \in N_1} T(b_1 n v_1) = (q^3 - q^2 \mp q) T_1 + (q^2 \pm q) T_0. \tag{23}$$

Case 2. Consider the second term of (21). The element $b_2 n(x_1, 0, 0, x_4) v_1$ is equal to

$${}^t(\boldsymbol{\pi}^r/2, x_1 \boldsymbol{\pi}^{-(r+1)}, 0, 0, \boldsymbol{\pi}^{-r}), \text{ or } {}^t(2\theta\boldsymbol{\pi}^r - x_1, \boldsymbol{\pi}^{-1}(x_1 \boldsymbol{\pi}^{-r} - x_4^2/2), x_4, \boldsymbol{\pi}, \boldsymbol{\pi}^{-r})$$

in the split and non-split cases respectively.

Let $r \geq 0$ in the split cases or $r \geq 1$ in a non-split case. We have:

(i) If $n = n(0,0,0,x_4) \in N_2$ the element $b_2 n d_r v_1$ is in the K-orbit of $d_r v_1$. Thus, $T(b_2 n d_r v_1) = T(d_r v_1) = T_r$.

(ii) If $n = n(x_1, 0, 0, x_4) \in N_2$ has $x_1 \neq 0$, the element $b_2 n d_r v_1$ is in the K-orbit of $d_{r+1} v_1$ and $T(b_2 n d_r v_1) = T(d_{r+1} v_1) = T_{r+1}$.

Since there are q matrices in *(i)* and $q^2 - q$ in *(ii)*, (for $r \geq 0$), we have

$$\sum_{n \in N_2} T(b_2 n d_r v_1) = (q^2 - q) T_{r+1} + q T_r. \tag{24}$$

Let $r = 0$. The element $b_2 n v_1$ is in the K-orbit of $d_1 v_1$ if $x_1 - \frac{1}{2} x_4^2 \neq 0$ ($q^2 - q$ cases), otherwise it is in the K-orbit of v_1. The contribution is the same as (24).

Case 3. Consider the third term of (21).

Let $r \geq 0$. The element $b_2^{-1} n(0,0,x_3,0) v_1$ is equal to

$${}^t(\boldsymbol{\pi}^r/2, 0, 0, x_3 \boldsymbol{\pi}^{-(r+1)}, \boldsymbol{\pi}^{-r}), \text{ or } {}^t(2\theta\boldsymbol{\pi}^r, 0, 0, \boldsymbol{\pi}^{-1}(1 + x_3 \boldsymbol{\pi}^{-r}) \boldsymbol{\pi}^{-r})$$

in the split and non-split cases respectively. This element belongs to the K-orbit of $d_r v_1$ if n is the identity matrix, and is in the K-orbit of $d_{r+1}v_1$ in the remaining $q-1$ cases. So, (for $r \geq 0$), we have

$$\sum_{n \in N_3} T(b_2^{-1} n d_r v_1) = (q-1)T_{r+1} + T_r. \tag{25}$$

Case 4. Since $b_1^{-1} d_r = d_{r+1}$ ($r \geq 0$), the last summand of (21) is

$$T(b_1^{-1} d_r v_1) = T_{r+1}. \tag{26}$$

Adding (22), (24), (25) and (26) we obtain that (when $r \geq 1$), the sum (21) is equal to

$$q^3 T_{r+1} + (q^2+q)T_r + T_{r-1}.$$

When $r=0$, adding (23), (24), (25) and (26) (used with $r=0$), the sum (21) is

$$(q^3 \mp q)T_1 + (q^2+q \pm q+1)T_0,$$

where the upper sign occurs in the split case and the lower in the non-split case. Both expressions should be equal to the right hand side of (20). Thus, we obtained two difference equations

$$q^3 T_{r+1} + (q^2+q)T_r + T_{r-1} = q^{\frac{3}{2}}(q^{\zeta} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\zeta})T_r,$$

and

$$(q^3 \mp q)T_1 + (q^2+q \pm q+1)T_0 = q^{\frac{3}{2}}(q^{\zeta} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\zeta})T_0.$$

The first one can be simplified to

$$q^3 T_{r+1} - q^{\frac{3}{2}}(q^{\zeta} + q^{-\zeta})T_r + T_{r-1} = 0 \tag{27}$$

In the second one we assume $T_0 = 1$. Then

$$T_1 = \frac{q^{\frac{3}{2}}(q^{\zeta} + q^{-\zeta}) \mp q - 1}{q^3 \mp q}. \tag{28}$$

The general solution of (27) is given by

$$T_r = c_1 \lambda_1^r + c_2 \lambda_2^r,$$

where λ_1 and λ_2 are the two roots of the quadratic equation

$$q^3 \lambda^2 - q^{\frac{3}{2}}(q^{\zeta} + q^{-\zeta})\lambda + 1 = 0, \tag{29}$$

and c_1, c_2 are chosen to satisfy two initial conditions:

$$1 = c_1 + c_2, \text{ and } T_1 = c_1\lambda_1 + c_2\lambda_2. \tag{30}$$

The solutions of (29) are $\lambda_1 = q^{-\frac{3}{2}+\zeta}$, $\lambda_2 = q^{-\frac{3}{2}-\zeta}$ and that of (30) are

$$c_1 = \frac{T_1 - \lambda_2}{\lambda_1 - \lambda_2} = \frac{(q^{\frac{3}{2}+\zeta} - 1)(1 \mp q^{-\frac{1}{2}-\zeta})}{(q^{\zeta} - q^{-\zeta})(q^{\frac{3}{2}} \mp q^{-\frac{1}{2}})},$$

$$c_2 = \frac{\lambda_1 - T_1}{\lambda_1 - \lambda_2} = -\frac{(q^{\frac{3}{2}-\zeta} - 1)(1 \mp q^{-\frac{1}{2}+\zeta})}{(q^{\zeta} - q^{-\zeta})(q^{\frac{3}{2}} \mp q^{-\frac{1}{2}})}.$$

The proposition follows. □

The main result of this subsection is:

Proposition 2.3. *Set $X = q^{\zeta}$. Then*

$$\int_S \Phi_r(s) T_{\zeta}(s) ds = q^r \left[q^{\frac{1}{2}r} \frac{X^{r+1} - X^{-(r+1)}}{X - X^{-1}} \mp q^{\frac{1}{2}(r-1)} \frac{X^r - X^{-r}}{X - X^{-1}} \right],$$

where the "−" sign is in the split case and the "+" in the non-split case.

Proof. We have

$$\int_S \Phi_r(s) T(s) ds = \int_S \sum_{k=0}^{r} \phi_k(s) T(s) ds = \sum_{k=0}^{r} T_k \Lambda_k. \tag{31}$$

Recall that $T_0 = 1$ and assume that $\Lambda_0 = 1$. In the computations below, the upper sign occurs in the split case and the lower in the non-split case. Using Propositions 1.4 and 2.2, we have that, the sum (31) is equal to

$$1 + \frac{(q^{\frac{3}{2}}X - 1)(1 \mp q^{-\frac{1}{2}}X^{-1})}{q^{\frac{3}{2}}(X - X^{-1})} \sum_{k=1}^{r} q^{\frac{3}{2}k} X^k - \frac{(q^{\frac{3}{2}}X^{-1} - 1)(1 \mp q^{-\frac{1}{2}}X)}{q^{\frac{3}{2}}(X - X^{-1})} \sum_{k=1}^{r} q^{\frac{3}{2}k} X^{-k}.$$

Using the summation formula for geometric series, and then simplifying the result we obtain the formula claimed in the proposition. □

II.2. The group H. Recall that $H = PGL(2, F)$, and $\pi'_{\zeta} = I_{H,\chi_0}(\zeta, -\zeta)$ denotes the representation of H, induced from the character $\mathrm{diag}(a, 1)n \mapsto |a|^{\zeta}\chi_0(a)$ of B', where χ_0 is 1 or the unramified character of $F^{\times}$, whose square is 1. We denote the elements of $PGL(2, F)$ by their representatives in $GL(2, F)$. The Bruhat decomposition of H is $H = B' \cup N'wB'$, $B' = N'A'$, where $A' = \{\mathrm{diag}(a, 1); a \in F^{\times}\}$,

$$N' = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix};\ x \in F \right\},\ w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The character ψ' of N' is defined by $\psi_{N'}(n(x)) = \psi(x)$.

Let W_{ζ} be the normalized unramified Whittaker function in the space of representation

π'_ζ. It satisfies $W_\zeta(e)=1$, $W_\zeta(ngk)=\psi_{N'}(n)W_\zeta(g)$ ($n'\in N'$, $k\in K'=PGL(2,R)$, the standard maximal compact subgroup of H), and for any $f'\in C_c^\infty(K'\backslash H/K')$, also

$$f'^\vee(\pi'_\zeta)W_\zeta(h)=\int_H f'(x)W_\zeta(hx)dx.$$

In particular, when $h=e$ we have

$$f'^\vee(\pi'_\zeta)=\int_H f'(x)W_\zeta(x)dx. \tag{32}$$

Since $dg=|a|^{-1}dndadk$, where $a=\mathrm{diag}(\alpha,1)$, $|a|=|\alpha|$ and $da=d^\times\alpha$, we obtain

$$f'^\vee(\pi'_\zeta)=\int_H f'(nak)W_\zeta(nak)|a|^{-1}dndadk$$

$$=\int_{A'}\int_{N'}f'(na)\psi_{N'}(n)dn\,W_\zeta(a)|a|^{-1}da=\int_{A'}\phi'_{f'}(a)W_\zeta(a)|a|^{-1}da,$$

where

$$\phi'_{f'}(g)=\int_{N'}f'(ng)\psi_{N'}(n)dn.$$

The function $\phi'_{f'}$ lies in the space $C_c(N'\backslash H/K',\psi_{N'})$ of the right K'-invariant, compactly supported modulo N', functions ϕ' on H, which satisfy $\phi'(ng)=\overline{\psi}_{N'}(n)\phi'(g)$. For any integer $r\geq 0$, define ϕ'_r by

$$\phi'_r\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)=\begin{cases}1, & \text{if } |\alpha|=q^{-r},\\ 0, & \text{otherwise .}\end{cases}$$

Proposition 2.4.

(1) The set $\{\phi'_r; r\geq 0\}$ is a basis of $C_c(N'\backslash H/K',\psi_{N'})$.

(2) For any $\phi'\in C_c(N'\backslash H/K',\psi_{N'})$, we have $\phi'(\mathrm{diag}(\alpha,1))=0$ if $|\alpha|>1$.

(3) As in Proposition 2.3, put $X=q^\zeta$. Then

$$\int_{A'}\phi'_r(a)W_\zeta(a)|a|^{-1}da=(-1)^r q^{\frac{1}{2}r}\frac{X^{r+1}-X^{-(r+1)}}{X-X^{-1}}.$$

Proof. (1) This is clear.

(2) Indeed, choosing $n\in F^\times$ such that $|n|=|\alpha|$, we have

$$\overline{\psi}_{N'}(n)\phi'\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)=\phi'\left(\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)=\phi'\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \alpha/n\\ 0 & 1\end{pmatrix}\right).$$

This is $\phi'(\mathrm{diag}(\alpha,1))$, since ϕ' is right K' invariant. But $\psi_{N'}$ has conductor R, hence $\psi_{N'}(n)\neq 1$ for some n, and our claim follows.

(3) This follows from the definition of ϕ'_r and Shintani's explicit formula [Sh] for the Whittaker function (cf. [F], p. 305) of $I_{H,\chi_0}(\zeta,-\zeta)$, which asserts that

$$W_\zeta\left(\begin{pmatrix} \boldsymbol{\pi}^r & 0 \\ 0 & 1 \end{pmatrix}\right) = (-1)^r q^{-\frac{1}{2}r}\frac{X^{r+1}-X^{-(r+1)}}{X-X^{-1}}.$$

□

II.3. The correspondence. Put $\pi_\zeta = I_G(\zeta, 1/2+\zeta)$ and $\pi'_\zeta = I_{H,\chi_0}(\zeta,-\zeta)$. Following [FM], we say that $f \in C(K\backslash G/K)$ and $f' \in C(K'\backslash H/K')$ are *corresponding* if $f^\vee(\pi_\zeta) = f'^\vee(\pi'_\zeta)$. By (18) and (32), an equivalent definition is given by

$$\int_S \phi_f(s)T_\zeta(s)ds = \int_{A'} \phi'_{f'}(a)W_\zeta(a)|a|^{-1}da.$$

Definition. Define a map $\mathcal{F} : C_c^\infty(K\backslash S) \to C_c(N'\backslash H/K', \psi_{N'})$ by $\mathcal{F}(\phi) = \phi'$ if

$$\int_S \phi(s)T_\zeta(s)ds = \int_{A'} \phi'(a)W_\zeta(a)|a|^{-1}da,$$

where T_ζ is the K-invariant function on S, defined in II.1 and Proposition 2.2, and W_ζ is the unramified normalized Whittaker function defined in Section II.2.

Proposition 2.5. *The map $\mathcal{F}$ is well defined and induces a linear bijection between the spaces $C_c^\infty(K\backslash S)$ and $C_c(N'\backslash H/K', \psi_{N'})$, given by the correspondence ($r \geq 0$)*

$$\mathcal{F}(\Phi_r) = (-1)^r q^r(\phi'_r \pm \phi'_{r-1}),$$

where the "+" sign occurs in the split case and the "−" sign in the non-split case.

Proof. This follows from Propositions 2.3 and 2.4. □

Corollary. If f and f' are corresponding spherical functions, then $\mathcal{F}(\phi_f) = \phi'_{f'}$.

III. The Fourier coefficients of orbital integrals.

III.1. The Fourier coefficients of orbital integrals on H. As usual, F is a p-adic field, and χ is a complex valued character of $F^\times$ with conductor m, $m \geq 0$. Thus if $m \geq 1$, this character is trivial on $1+\boldsymbol{\pi}^m R$ and is non-trivial on $1+\boldsymbol{\pi}^{m-1}R$. If $m=0$ then χ is trivial on $R^\times$ and is non-trivial on $1+\boldsymbol{\pi}^{-1}R$, and we say that χ is unramified. Recall that $\{\phi'_r; r \geq 0\}$ is the basis of $C_c(N'\backslash H/K', \psi_{N'})$, and for any $\phi' \in C_c(N'\backslash H/K', \psi_{N'})$, we defined

$$\Psi'(\alpha, \phi') = \int_{F^\times} \phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\chi_0(a)d^\times a. \tag{33}$$

Definition. For any $r \geq 0$, set

$$\widehat{\Psi}'_r(\chi) = \int_{F^\times} (\theta, \alpha)\psi(\alpha)\chi(\alpha)|\alpha|\Psi'(\alpha^{-1}, \phi'_r)d^\times\alpha. \tag{34}$$

Proposition 3.1. *Let χ be a multiplicative character of $F^\times$. If χ is unramified, define ζ and X by $X = |\boldsymbol{\pi}|^{-\zeta} = \chi(\boldsymbol{\pi})^{-1}$. Then*

$$\int_{F^\times} (\theta, \alpha)\psi(\alpha)\chi(\alpha)|\alpha|\Psi'(\alpha^{-1}, (-1)^r q^r(\phi'_r \pm \phi'_{r-1}))d^\times\alpha$$

is equal to 0 *if χ is ramified, and is equal to*

$$X^{-r} \pm qX^{1-r}$$

if χ is unramified.

This proposition follows from the following Proposition :

Proposition 3.2. *The integral*

$$\widehat{\Psi}'_r(\chi) = \int_{F^\times} (\theta, \alpha)\psi(\alpha)\chi(\alpha)|\alpha|\Psi'(\alpha^{-1}, \phi'_r)d^\times\alpha$$

is equal to

$$(-1)^r(qX)^{-r}\frac{1 \pm qX}{1 \mp qX} + (\mp 1)^{r-1}\frac{2q^2X(1 \mp X)}{(q-1)(1 \mp qX)}, \tag{35}$$

in the split and the non-split cases respectively, if χ is unramified, and to

$$2(-1)^r q^m \tau(\psi, \chi), \quad \textit{where} \quad \tau(\psi, \chi) = \int_{|x|=q^m} \psi(x)\chi(x)d^\times x$$

if χ is ramified with conductor m.

Let us show how Proposition 3.1 follows from Proposition 3.2:

Proof of Proposition 3.1. If χ is ramified, the result is obvious. If χ is unramified, using the result of Proposition 3.2, we have that

$$(-1)^r q^r(\widehat{\Psi}'_r(\chi) \pm \widehat{\Psi}'_{r-1}(\chi))$$

is equal to (in the split/non-split cases)

$$X^{-r}\frac{1 \pm qX \mp (qX \pm (qX)^2)}{1 \mp qX} = \frac{1-(qX)^2}{1 \mp qX}X^{-r} = X^{-r} \pm qX^{1-r}.$$

$\square$

To prove Proposition 3.2, we need the following self contained Lemmas:

Lemma 3.3. *Let ψ be a character of F with conductor R and χ an unramified character of $F^\times$, where F is p-adic field. Set $|\boldsymbol{\pi}| = q^{-1}$ and $\chi(\boldsymbol{\pi})^{-1} = X$. Then for any $x \in F^\times$, we have*

$$\int_{|\alpha|=q^k} |\alpha|^3 \psi(\alpha x)\chi(\alpha) d^\times \alpha = \begin{cases} 0, & \textit{if } \ q^k \geq q^2|x|^{-1}, \\ -(q-1)^{-1}(q^3X)^k, & \textit{if } \ q^k = q|x|^{-1}, \\ (q^3X)^k, & \textit{if } \ q^k \leq |x|^{-1}. \end{cases} \tag{36}$$

Proof. Recalling that $d^\times \alpha = (1-1/q)^{-1}|\alpha|^{-1}d\alpha$, and that if $|\alpha| = q^k$ then $\chi(\alpha) = X^k$, we obtain

$$\int_{|\alpha|=q^k} |\alpha|^3 \psi(\alpha x)\chi(\alpha) d^\times \alpha = \left(1 - \frac{1}{q}\right)^{-1} q^{2k} X^k \int_{|\alpha|=q^k} \psi(\alpha x) d\alpha.$$

Put $\beta = \alpha x$. The latter integral is

$$\left(1 - \frac{1}{q}\right)^{-1} q^{2k} X^k |x|^{-1} \int_{|\beta|=q^k|x|} \psi(\beta) d\beta.$$

Recall that

$$\int_{|\beta|=q^l} \psi(\beta) d\beta = \begin{cases} 0, & \text{if } \ l \geq 2, \\ -1, & \text{if } \ l = 1, \\ (1-1/q)q^l, & \text{if } \ l \leq 0. \end{cases}$$

The lemma follows from this if we note that if $l = 1$, i.e. $q^k|x| = q$, then

$$q^{2k} X^k |x|^{-1} = \frac{1}{q}(q^3X)^k.$$

The other cases are obvious. $\square$

Lemma 3.4. *The same notations as in Lemma 3.3, but let χ be a ramified character with conductor m, $m \geq 1$. Then*

$$\int_{|x|=q^k} \psi(x)\chi(x) d^\times x$$

is equal to 0 *unless $k = m$, in which case we denote it by $\tau(\psi, \chi)$.*

Proof. We will consider two cases: $k > m$ and $k < m$.

Case 1. Consider the case $k > m$. In this case, there exists an element $y \in F^\times$, such that $|y| > 1$, $1 + yx^{-1} \in 1 + \boldsymbol{\pi}^m R$ and $\psi(y) \neq 1$. For such y, we have

$$\chi(1 + y/x) = 1, \ |x+y| = |x|.$$

The change of variables $x \mapsto x + y$, gives

$$\int_{|x|=q^k} \psi(x)\chi(x) d^\times x = \int_{|x|=q^k} \psi(x+y)\chi(x(1+y/x)) d^\times x = \psi(y) \int_{|x|=q^k} \psi(x)\chi(x) d^\times x.$$

This is 0 since $\psi(y) \neq 1$.

Case 2. Consider the case $k < m$. In this case, take an element $y \in 1 + \boldsymbol{\pi}^m R$ (thus $\psi(y) = 1$, $|xy| = |x|$) such that $\chi(y) \neq 1$. Set $y' = y - 1$. Since $|xy'| \leq 1$, we have

$$\psi(xy) = \psi(x + xy') = \psi(xy')\psi(x) = \psi(x).$$

The change of variables $x \mapsto xy$, gives

$$\int_{|x|=q^k} \psi(x)\chi(x) d^\times x = \int_{|x|=q^k} \psi(xy)\chi(xy) d^\times x = \chi(y) \int_{|x|=q^k} \psi(x)\chi(x) d^\times x.$$

This is 0, since $\chi(y) \neq 1$. The lemma follows. □

Proof of Proposition 3.2. The integral $\widehat{\Psi}'_r(\chi)$ is given by the integral

$$\int_{F^\times} \int_{F^\times} \phi'_r \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) (\theta, \alpha) \chi(\alpha) \iota(a) \psi(\alpha) |\alpha| d^\times a d^\times \alpha, \tag{37}$$

where $\iota(a) = \chi_0(a)$ in the split case and is 1 in the non-split case. By matrix multiplication

$$\begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1}a^{-1} \\ 0 & 1 \end{pmatrix}. \tag{38}$$

We will consider two cases.

Case 1. Assume that $|\alpha^{-1}a^{-1}| \leq 1$ or $|\alpha| \geq |a|^{-1}$. Using (37) and the property of ϕ'_r

$$\phi'_r \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = \phi'_r \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = \phi'_r \left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \right).$$

In $PGL(2)$ this is equal to $\phi'_r \left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right)$, which is 0 unless $|a|^{-1} = q^{-r}$ or $|a| = q^r$. Hence, the integral (37) is equal to

$$\int_{|a|=q^r} \iota(a) d^\times a \int_{|\alpha| \geq q^{-r}} (\theta, \alpha) \chi(\alpha) \psi(\alpha) |\alpha| d^\times \alpha = \iota^r(\boldsymbol{\pi}) \sum_{l \geq -r} \int_{|\alpha|=q^l} (\theta, \alpha) \chi(\alpha) \psi(\alpha) |\alpha| d^\times \alpha. \tag{39}$$

If χ is unramified, we apply Lemma 3.3 with $|\alpha|$ instead of $|\alpha|^3$ and $x = 1$:

$$\int_{|\alpha|=q^l} \chi(\alpha)\psi(\alpha)|\alpha| d^\times \alpha = \begin{cases} 0, & \text{if } \ l \geq 2, \\ -X(1-1/q)^{-1}, & \text{if } l = 1, \\ (qX)^l, & \text{if } l \leq 0. \end{cases} \tag{40}$$

Futhermore, in the split case $(\theta,\alpha)=1$, and in the non-split case $(\theta,\alpha)=(-1)^l$ if $|\alpha|=q^l$. So, the sum (39) is equal to

$$(\mp 1)^r\sum_{l=-r}^{0}(\pm qX)^l+(\mp 1)^{r-1}\frac{qX}{q-1}=\frac{(-1)^r(qX)^{-r}+(\mp 1)^{r-1}qX}{1\mp qX}+(\mp 1)^{r-1}\frac{qX}{q-1}. \quad (41)$$

If χ is ramified with conductor m, then according to Lemma 3.4, the integral (39) is equal to

$$(-1)^r q^m\int_{|\alpha|=q^m}\chi(\alpha)\psi(\alpha)d^\times\alpha=(-1)^r q^m\tau(\psi,\chi).$$

Case 2. In this case, $|\alpha|<|a|^{-1}$. Note that

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&\alpha^{-1}\\0&1\end{pmatrix}\begin{pmatrix}a&0\\0&1\end{pmatrix}=\begin{pmatrix}0&1\\-a&-\alpha^{-1}\end{pmatrix}.$$

In $PGL(2)$, we have

$$\begin{pmatrix}0&1\\-a&-\alpha^{-1}\end{pmatrix}\begin{pmatrix}-\alpha&0\\0&-\alpha\end{pmatrix}=\begin{pmatrix}0&-\alpha\\a\alpha&1\end{pmatrix}=\begin{pmatrix}1&-\alpha\\0&1\end{pmatrix}\begin{pmatrix}a\alpha^2&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\a\alpha&1\end{pmatrix}.$$

Hence,

$$\phi_r'\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&\alpha^{-1}\\0&1\end{pmatrix}\begin{pmatrix}a&0\\0&1\end{pmatrix}\right)=\psi(\alpha)\phi_r'\left(\begin{pmatrix}a\alpha^2&0\\0&1\end{pmatrix}\right).$$

The integral (37) reduces to

$$\int_{F^\times}\int_{F^\times}\psi(\alpha)\phi_r'\left(\begin{pmatrix}a\alpha^2&0\\0&1\end{pmatrix}\right)\iota(a)(\theta,\alpha)\chi(\alpha)\psi(\alpha)|\alpha|d^\times\alpha d^\times a.$$

This integral does not vanish precisely when $|a\alpha^2|=q^{-r}$. Thus, it is

$$\int_{|\alpha|<|a|^{-1}}\int_{|a|=q^{-r}|\alpha|^{-2}}\iota(a)(\theta,\alpha)\chi(\alpha)\psi(\alpha)|\alpha|d^\times\alpha d^\times a. \quad (42)$$

Set $|\alpha|=q^l$ and $|a|=q^s$. The above integral is taken over the set $l<-s$, $s+2l=-r$. Equivalently, this set is defined by $l>-r$, $s=-r-2l$. Applying (40) to (42), the integral is a finite sum (split/non-split cases respectively)

$$\iota^r(\boldsymbol{\pi})\sum_{l>-r}\int_{|\alpha|=q^l}(\theta,\alpha)\chi(\alpha)\psi(\alpha)|\alpha|d^\times\alpha=(\mp 1)^r\sum_{l=1-r}^{0}(\pm qX)^l+(\mp 1)^{r-1}\frac{qX}{q-1}$$

if χ is unramified, and to $(-1)^r q^m\tau(\psi,\chi)$ if χ is ramified. Using the summation formula for geometric series, the unramified case is

$$(\mp 1)^r\frac{(\pm qX)^{-r}-1}{(\pm qX)^{-1}-1}+(\mp 1)^{r-1}\frac{qX}{q-1}=\frac{(\mp 1)^r(\pm qX)^{1-r}\mp qX}{1\mp qX}+(\mp 1)^{r-1}\frac{qX}{q-1}.$$

Combining this expression with (41), we obtain the final result (for the unramified case)

$$\frac{(-1)^r(qX)^{-r}+(\mp 1)^{r-1}qX}{1\mp qX}+\frac{(-1)^{r-1}(qX)^{1-r}+(\mp 1)^{r-1}qX}{1\mp qX}-\frac{2qX}{q-1}.$$

Once simplified, it completes the proof of the Proposition 3.2. □

III.2. The Fourier coefficients of orbital integrals on G. Recall that for any $\phi\in C_c^\infty(K\backslash S)$ and $a_\alpha=\mathrm{diag}(\alpha,1,1,1,\alpha^{-1})$, we defined

$$\Psi(\alpha,\phi)=\int_N \phi(na_\alpha\gamma_0 v_0)\psi_N(n)dn.$$

For any $\phi\in C_c(K\backslash S)$, the Fourier transform $\widehat{\Psi}_\phi(\chi)$ of $\Psi(\alpha,\phi)$ is given by

$$\widehat{\Psi}_\phi(\chi)=\int_{F^\times}\Psi(\alpha,\phi)\chi(\alpha)d^\times\alpha.$$

Recall that $\Phi_r=\sum_{i=0}^r\phi_i$, where ϕ_i is the characteristic function of the K-orbit of d_iv_1. In this section we compute

$$\widehat{\Psi}_r(\chi)=\widehat{\Psi}_{\Phi_r}(\chi),$$

where χ is the same character as in Section II.

Proposition 3.5. *In the split case, the Fourier transform $\widehat{\Psi}_r(\chi)$ of $\Psi(\alpha,\Phi_r)$ is equal to*

$$qX^{1-r}+X^{-r}$$

if the character χ is unramified, and to 0 if χ is ramified.

Proof. Recalling the definitions of a, n, γ_0 and v_0, we have

$$\Psi(\alpha,\Phi_r)=\iiint_{F^3}\Phi_r(\alpha/2+z/\alpha,x_1/\alpha,x_2/\alpha,x_3/\alpha,1/\alpha)\psi(x_2)dx_1dx_2dx_3.$$

Recall that $\Phi_r(x_1,x_2,x_3,x_4,x_5)$ is 1 if $|x_i|\le q^r$ $(i=1,...,5)$ and is zero otherwise. Thus, the integral $\widehat{\Psi}_r(\chi)$ is equal to

$$\iiiint \chi(\alpha)\psi(x_2)dx_1dx_2dx_3d^\times\alpha, \tag{43}$$

over the set defined by

$$\left|\frac{\alpha}{2}+\frac{z}{\alpha}\right|\le q^r,\ \left|\frac{x_1}{\alpha}\right|\le q^r,\ \left|\frac{x_2}{\alpha}\right|\le q^r,\ \left|\frac{x_3}{\alpha}\right|\le q^r,\ \left|\frac{1}{\alpha}\right|\le q^r,$$

where $z = -x_1x_3 - x_2^2/2$. After the change of variables $x_i \mapsto \alpha x_i$, the integral (43) is equal to

$$\iiiint |\alpha|^3 \chi(\alpha)\psi(\alpha x_2) dx_1 dx_2 dx_3 d^\times \alpha, \tag{44}$$

over the set

$$|\alpha(1 - x_2^2 - 2x_1x_3)| \le q^r,\ |x_1| \le q^r,\ |x_2| \le q^r,\ |x_3| \le q^r,\ q^{-r} \le |\alpha|.$$

Fixing x_1, x_2 and x_3, we obtain that

$$q^{-r} \le |\alpha| \le q^r |1 - x_2^2 - 2x_1x_3|^{-1}.$$

Changing the order of integration in (44), it is equal to

$$\int_{|x_1|\le q^r} \int_{|x_3|\le q^r} \int_{|x_2|\le q^r} \int_{q^{-r}\le|\alpha|\le q^r|1-x_2^2-2x_1x_3|^{-1}} |\alpha|^3 \chi(\alpha)\psi(\alpha x_2) d^\times \alpha dx_2 dx_3 dx_1. \tag{45}$$

First, assume that χ is ramified with conductor m. By Lemma 3.4

$$\int_{|\alpha|=q^k} \chi(\alpha)\psi(\alpha) d^\times \alpha$$

vanishes unless $k = m$. Thus

$$\int_{|\alpha|=q^k} |\alpha|^3 \chi(\alpha)\psi(\alpha x_2) d^\times \alpha = q^{3m} \tau(\psi, \chi) |x_2|^{-3} \chi^{-1}(x_2).$$

Since, for any l, the integral of $\chi^{-1}(x_2)$ over $|x_2| = q^l$ is equal to 0, we conclude that $\widehat{\Psi}_r(\chi) = 0$ when χ is ramified.

Now, let us consider the case of an unramified character χ. To apply Lemma 3.3, we need to split the domain of integration over α into two domains: defined by condition $q^r/|1 - 2x_1x_3 - x_2^2|$ is $\le 1/|x_2|$ or $> 1/|x_2|$. We will consider the contributions of the integral (45) over each of these domains. There are two cases.

Case 1. Let us consider the contribution to (45) from the first domain, namely $q^r/|1 - 2x_1x_3 - x_2^2| \le 1/|x_2|$. Equivalently, it is

$$\left| \frac{1 - 2x_1x_3}{x_2} - x_2 \right| \ge q^r. \tag{46}$$

In this case there are two subdomains.

Case 1a. The first subdomain is $|x_2| = q^r$. Since $|x_i| \le q^r$, we have $|1 - 2x_1x_3| \le q^{2r}$, which implies

$$\left| \frac{1 - 2x_1x_3}{x_2} - x_2 \right| = q^r.$$

This is equivalent to $|1-2x_1x_3-x_2^2| = |x_2|q^r = q^{2r}$. Note that since $r \geq 1$, we have $|1-2x_1x_3-x_2^2| = |2x_1x_3+x_2^2|$. Hence, we conclude that in the integral (45), the integration over α is taken over $|\alpha| = q^{-r}$. Applying Lemma 3.3,

$$\int_{|\alpha|=q^{-r}} |\alpha|^3\chi(\alpha)\psi(\alpha x_2)d^\times\alpha = (q^3X)^{-r}.$$

The integral (45) is the product of $(q^3X)^{-r}$ and the volume of a subset defined by

$$\{|x_1| \leq q^r, |x_2| \leq q^r, |x_3| \leq q^r, |2x_1x_3 + x_2^2| = q^{2r}\}.$$

This subset is equal to

$$\{|x_1x_3| \leq q^{2r-1}, |x_2| = q^r\} \cup \{|x_1| = q^r, |x_3| = q^r, |x_2| = q^r, |2x_1x_3 + x_2^2| = q^{2r}\}. \tag{47}$$

The volume of the first subset is

$$\left[\int_{|x_1|<q^r}\int_{|x_3|\leq q^r} + \int_{|x_1|=q^r}\int_{|x_3|<q^r}\right] dx_3dx_1 \int_{|x_2|=q^r} dx_2 = \Big(1-\frac{1}{q}\Big)\Big(2-\frac{1}{q}\Big)q^{3r-1}.$$

The volume of the second subset of (47) is the integral

$$\int_{|x_1|=q^r}\int_{|x_2|=q^r}\int_{|x_3|=q^r,\,|2x_1x_3+x_2^2|=q^{2r}} dx_3dx_2dx_1 = \Big(1-\frac{1}{q}\Big)^2\Big(1-\frac{2}{q}\Big)q^{3r}.$$

Multiplying the volume of (47) by $(q^3X)^{-r}$, the contribution of (45) from the subcase 1a is:

$$\Big(1-\frac{1}{q}\Big)\Big(2-\frac{1}{q}\Big)\frac{1}{q}X^{-r} + \Big(1-\frac{1}{q}\Big)^2\Big(1-\frac{2}{q}\Big)X^{-r}. \tag{48}$$

Case 1b. The second subdomain is defined by $|x_2| < q^r$. In this case, we have

$$\left|\frac{1-2x_1x_3}{x_2} - x_2\right| = \left|\frac{1-2x_1x_3}{x_2}\right|.$$

Hence, in the integral (45), the integration over α is performed over

$$q^{-r} \leq |\alpha| \leq \frac{q^r}{|1-2x_1x_3|},$$

and x_2 satisfies $\{|x_2| < q^r, |x_2| \leq |1-2x_1x_3|q^{-r}\}$. We will consider two cases: $|1-2x_1x_3| = q^{2r}$ and $|1-2x_1x_3| < q^{2r}$.

(i) Let $|1-2x_1x_3| = q^{2r}$. Since $|x_i| \leq q^r$, this implies that $|x_1| = |x_3| = q^r$. The integration (in (45)) is taken only over α with $|\alpha| = q^{-r}$, and over x_2 with $|x_2| < q^r$. Thus the integral (45) is

$$\int_{|x_1|=q^r}\int_{|x_3|=q^r}\int_{|x_2|<q^r}\int_{|\alpha|=q^{-r}} |\alpha|^3\chi(\alpha)\psi(\alpha x_2)d^\times\alpha dx_2dx_3dx_1.$$

Once evaluated and simplified, it is

$$\Big(1-\frac{1}{q}\Big)^2 q^{2r}q^r\frac{1}{q}(q^3X)^{-r} = \Big(1-\frac{1}{q}\Big)^2\frac{1}{q}X^{-r}. \tag{49}$$

(ii) Let $|1-2x_1x_3| < q^{2r}$. Define l by $|1-2x_1x_3| = q^l$. In (45), x_2 is bounded from above by $|1-2x_1x_3|q^{-r} = q^{l-r}$. Since $l \leq 2r-1$, this implies that $x_2 < q^r$. The integral (45) becomes

$$\int_{|x_1|\leq q^r}\int_{|x_3|\leq q^r}\int_{|x_2|\leq|1-2x_1x_3|q^{-r}}\int_{q^{-r}\leq|\alpha|\leq q^r|1-2x_1x_3|^{-1}}|\alpha|^3\chi(\alpha)\psi(\alpha x_2)d^\times\alpha dx_2dx_3dx_1.$$

Breaking it into the sum over l, it is

$$\sum_{l\leq 2r-1}\iint_{|x_1|\leq q^r,|x_3|\leq q^r,\ |1-2x_1x_3|=q^l}\int_{|x_2|\leq q^{l-r}}\int_{q^{-r}\leq|\alpha|\leq q^{r-l}}|\alpha|^3\chi(\alpha)\psi(\alpha x_2)d^\times\alpha dx_2dx_1dx_3. \tag{50}$$

Applying Lemma 3.3, the integral over α becomes a geometric series $\sum_{k=-r}^{r-l}q^{3k}X^k$. Substituting this into the integral over x_2 in (50), we obtain

$$\int_{|x_2|\leq q^{l-r}}\frac{(q^3X)^{r-l+1}-(q^3X)^{-r}}{q^3X-1}dx_2 = \frac{q(q^2X)^{r+1}}{q^3X-1}(q^2X)^{-l}-\frac{(q^4X)^{-r}}{q^3X-1}q^l.$$

Hence, the integral (50) is equal to

$$\sum_{l\leq 2r-1}\iint_{|x_1|\leq q^r,\,|x_3|\leq q^r,\,|1-2x_1x_3|=q^l}\left[\frac{q(q^2X)^{r+1}}{q^3X-1}(q^2X)^{-l}-\frac{(q^4X)^{-r}}{q^3X-1}q^l\right]dx_1dx_3.$$

Splitting off the term corresponding to $l = 2r-1$, it is

$$\frac{1}{q}(q^3X+1)(q^2X)^{-r}\iint_{|x_1|\leq q^r,\,|x_3|\leq q^r,\,|1-2x_1x_3|=q^{2r-1}}dx_1dx_3 \tag{51}$$

$$+\sum_{l\leq 2r-2}\left[\frac{q(q^2X)^{r+1}}{q^3X-1}(q^2X)^{-l}-\frac{(q^4X)^{-r}}{q^3X-1}q^l\right]\iint_{|x_1|\leq q^r,\,|x_3|\leq q^r,\,|1-2x_1x_3|=q^l}dx_1dx_3. \tag{52}$$

This is a contribution of (45) over the domain 1b(ii). We will not evaluate the sum (52) any further for it will be cancelled by a similar sum obtained below.

Case 2. Consider the contribution of (45) over the domain $q^r/|1-2x_1x_3-x_2^2| > 1/|x_2|$. Equivalently, it is

$$\left|\frac{1-2x_1x_3}{x_2}-x_2\right| < q^r. \tag{53}$$

Using Lemma 3.3, the integral over α in (45) can be split into two integrals: one over the subdomain $q^{-r}\leq|\alpha|\leq|x_2|^{-1}$ and the other one over the subdomain $|\alpha| = q|x_2|^{-1}$. Thus the integral (45) is

$$\iiint\left[\int_{q^{-r}\leq|\alpha|\leq|x_2|^{-1}}+\int_{|\alpha|=q|x_2|^{-1}}\right]|\alpha|^3\psi(\alpha x_2)\chi(\alpha)d^\times\alpha dx_1dx_2dx_3, \tag{54}$$

where x_1, x_2 and x_3 range over the set

$$|x_1| \leq q^r,\ |x_3| \leq q^r,\ \left|\frac{1-2x_1x_3}{x_2} - x_2\right| < q^r. \tag{55}$$

Define l by $|1-2x_1x_3| = q^l$. We distinguish between the cases $l \leq 2r-1$ and $l = 2r$.

Case 2a. Assume that $|1-2x_1x_3| = q^l < q^{2r}$, or $l \leq 2r-1$. Since $|x_2| \leq q^r$, the condition (53) cannot be satisfied for $l = 2r-1$. Hence $l \leq 2r-2$. Furthermore, (53) implies that $|1-2x_1x_3|q^{-r} < |x_2| < q^r$. Hence, the set (55) is

$$|x_1| \leq q^r,\ |x_3| \leq q^r,\ \frac{|1-2x_1x_3|}{q^{r-1}} \leq |x_2| \leq q^{r-1}. \tag{56}$$

Set $|x_2| = q^{r_1}$. By Lemma 3.3, we have

$$\int\limits_{q^{-r} \leq |\alpha| \leq q^{-r_1}} |\alpha|^3 \psi(\alpha x_2)\chi(\alpha) d^\times \alpha = \sum_{k=-r}^{-r_1} (q^3X)^k = \frac{q^3X}{q^3X-1}(q^3X)^{-r_1} - \frac{(q^3X)^{-r}}{q^3X-1}.$$

The other integral over α in (54) is

$$\int\limits_{|\alpha| = q|x_2|^{-1}} |\alpha|^3 \psi(\alpha x_2)\chi(\alpha) d^\times \alpha = -\Big(1-\frac{1}{q}\Big)^{-1} q^2 X q^{-3r_1} X^{-r_1} = -\frac{1}{q-1} q^3 X (q^3X)^{-r_1}.$$

Summing up the last two integrals and simplifying the result, the integration in (54) over x_2 is

$$\int\limits_{q^{l-r+1} \leq |x_2| \leq q^{r-1}} \left[\frac{q^4X(1-q^2X)}{(q^3X-1)(q-1)}(q^3X)^{-r_1} - \frac{(q^3X)^{-r}}{q^3X-1}\right] dx_2.$$

This integral is equal to

$$\Big(1-\frac{1}{q}\Big)\frac{q^4X(1-q^2X)}{(q^3X-1)(q-1)} \sum_{r_1=l-r+1}^{r-1} \Big(\frac{1}{q^2X}\Big)^{r_1} - \Big(1-\frac{1}{q}\Big)\frac{(q^3X)^{-r}}{q^3X-1} \sum_{r_1=l-r+1}^{r-1} q^{r_1}.$$

Once simplified, it is equal to

$$\frac{1}{q}(q^3X+1)(q^2X)^{-r} - \frac{q(q^2X)^{r+1}}{q^3X-1}(q^2X)^{-l} + \frac{(q^4X)^{-r}}{q^3X-1}q^l.$$

Hence, the integral (54) is

$$\frac{1}{q}(q^3X+1)(q^2X)^{-r} \sum_{l \leq 2r-2} \iint_{|x_1| \leq q^r,\, |x_3| \leq q^r,\, |1-2x_1x_3| = q^l} dx_1 dx_3 \tag{57}$$

$$-\sum_{l\le 2r-2}\left[\frac{q(q^2X)^{r+1}}{q^3X-1}(q^2X)^{-l}-\frac{(q^4X)^{-r}}{q^3X-1}q^l\right]\iint_{|x_1|\le q^r,\,|x_3|\le q^r,\,|1-2x_1x_3|=q^l}dx_1dx_3. \tag{58}$$

The sums (52) and (58) cancel each other. Thus, the sum of the contributions of Case 1b(ii) and Case 2a is obtained on adding (51) and (57). It is

$$\frac{1}{q}(q^3X+1)(q^2X)^{-r}\sum_{l\le 2r-1}\iint_{|x_1|\le q^r,\,|x_3|\le q^r,\,|1-2x_1x_3|=q^l}dx_1dx_3.$$

The sum of the integrals in this expression is the integral

$$\iint_{|x_1|\le q^r,\,|x_3|\le q^r,\,|1-2x_1x_3|\le q^{2r-1}}dx_1dx_3$$

$$=\int_{|x_1|\le q^{r-1}}dx_1\int_{|x_3|\le q^r}dx_3+\int_{|x_1|=q^r}dx_1\int_{|x_3|\le q^{r-1}}dx_3=q^{2r-1}(2-1/q).$$

In conclusion, the contribution from Case 1b(ii) and Case 2a is

$$\frac{1}{q}(q^3X+1)(q^2X)^{-r}q^{2r-1}\Big(2-\frac{1}{q}\Big). \tag{59}$$

Case 2b. Now $|1-2x_1x_3|=q^{2r}$. Since $|x_i|\le q^r$, this implies that $|x_1|=|x_3|=q^r$. Futhermore, to satisfy (53), we must have that $|x_2|=q^r$. Since in this case the inequality $q^{-r}\le|\alpha|\le|x_2|^{-1}$ is equivalent to $|\alpha|=q^{-r}$, and $q|x_2|^{-1}=q^{1-r}$, the integral (54) is equal to

$$\iiint\left[\int_{|\alpha|=q^{-r}}+\int_{|\alpha|=q^{1-r}}\right]|\alpha|^3\psi(\alpha x_2)\chi(\alpha)d^\times\alpha dx_1dx_2dx_3. \tag{60}$$

where x_1, x_2 and x_3 range over the set

$$|x_1|=q^r,\ |x_2|=q^r,\ |x_3|=q^r,\ \left|\frac{1-2x_1x_3}{x_2}-x_2\right|\le q^{r-1}. \tag{61}$$

Since $|x_2|=q^r$, using Lemma 3.3, we have

$$\int_{|\alpha|=q^{-r}}|\alpha|^3\psi(\alpha x_2)\chi(\alpha)d^\times\alpha=(q^3X)^{-r}, \tag{62}$$

and

$$\int_{|\alpha|=q^{1-r}}|\alpha|^3\psi(\alpha x_2)\chi(\alpha)d^\times\alpha=-\frac{(q^3X)^{1-r}}{q-1}. \tag{63}$$

The volume of the set (61) can be computed as follows. The inequality (53) is equivalent to

$$1-2x_1x_3-x_2^2=\boldsymbol{\pi}^{1-r}tx_2,\quad \text{where}\ \ |t|\le 1.$$

Thus, the set (61) can be described as

$$|x_1| = q^r,\ |x_2| = q^r,\ x_3 = \frac{1 - x_2^2}{2x_1} - \frac{x_2 \boldsymbol{\pi}^{1-r}}{2x_1} t,\ |t| \le 1.$$

Note that $|x_3| = q^r$ and $dx_3 = q^{r-1} dt$. The volume of (61) is

$$q^{r-1} \int_{|x_1|=q^r} dx_1 \int_{|x_2|=q^r} dx_2 \int_{|t|\le 1} dt = \Big(1 - \frac{1}{q}\Big)^2 q^{3r-1}.$$

Multiplying this by the sum of (62) and (63), the integral (60) is

$$\left[(q^3X)^{-r} - \frac{(q^3X)^{1-r}}{q-1}\right] \Big(1 - \frac{1}{q}\Big)^2 q^{3r-1}.$$

Hence, the contribution to (45) of Case 2b is

$$\frac{1}{q}\Big(1 - \frac{1}{q}\Big)^2 X^{-r} - (q-1)X^{1-r}. \tag{64}$$

We have considered all possible cases. Finally, the answer (the integral (45)) is obtained on adding (48), (49), (59) and (64). $\square$

Proposition 3.6. *In the non-split case, the Fourier transform $\widehat{\Psi}_r(\chi)$ of $\Psi(\alpha, \Phi_r)$ is equal to*

$$X^{-r} - qX^{1-r}$$

if the character χ is unramified, and to 0 if χ is ramified.

Proof. Recalling the definitions of a, n, γ_0 and v_0, we have

$$\Psi(\alpha, \Phi_r) = \iiint_{F^3} \Phi_r(2\alpha\theta - x_1 + z/\alpha, x_1/\alpha, x_2/\alpha, 1 + x_3/\alpha, 1/\alpha)\psi(x_1 + 2\theta x_3) dx_1 dx_2 dx_3.$$

Hence, by definition the integral $\widehat{\Psi}_r(\chi)$ is

$$\int_{F^\times} \int_{F^3} \Phi_r(2\alpha\theta - x_1 + z/\alpha, x_1/\alpha, x_2/\alpha, 1 + x_3/\alpha, 1/\alpha)\psi(x_1 + 2\theta x_3) dx_1 dx_2 dx_3 \chi(\alpha) d^\times\alpha.$$

Recall that $\Phi_r(x_1, x_2, x_3, x_4, x_5)$ is 1 if $|x_i| \le q^r$ $(i = 1, ..., 5)$ and is zero otherwise. Thus, the integral above becomes

$$\iiiint \chi(\alpha)\psi(x_1 + 2\theta x_3) dx_1 dx_2 dx_3 d^\times\alpha,$$

over the set defined by

$$\left|2\alpha\theta - x_1 + \frac{z}{\alpha}\right| \le q^r,\ \left|\frac{x_1}{\alpha}\right| \le q^r,\ \left|\frac{x_2}{\alpha}\right| \le q^r,\ \left|1 + \frac{x_3}{\alpha}\right| \le q^r,\ \left|\frac{1}{\alpha}\right| \le q^r,$$

where $z = -x_1x_3 - x_2^2/2$. After the change of variables $x_i \mapsto \alpha x_i$ followed by $x_3 \mapsto 1 + x_3$ we arrive at the integral

$$\iiiint |\alpha|^3 \psi(\alpha(x_1 + 2\theta(x_3 - 1))\chi(\alpha) d^\times \alpha dx_1 dx_2 dx_3 \tag{65}$$

over the set given by

$$|x_1| \le q^r,\ |x_2| \le q^r,\ |x_3| \le q^r,\ q^{-r} \le |\alpha| \le q^r |2 + \theta z|^{-1}, \tag{66}$$

where $z = -x_1x_3 - x_2^2/2$. By Lemma 3.3

$$\int_{|\alpha| = q^k} |\alpha|^3 \psi(\alpha x) \chi(\alpha) d^\times \alpha = \begin{cases} 0, & \text{if } \ q^k \ge q^2 |x|^{-1}, \\ -(q-1)^{-1}(q^3X)^k, & \text{if } \ q^k = q|x|^{-1}, \\ (q^3X)^k, & \text{if } \ q^k \le |x|^{-1}. \end{cases} \tag{67}$$

In order to use this, we will split (66) into two subdomains: according to whether $q^r/|2+\theta z|$ is $\le$ or $>$ than $1/|x_1 + 2\theta(x_3 - 1)|$.

Case 1. We have $q^r/|2 + \theta z| \le 1/|x_1 + 2\theta(x_3 - 1)|$. Using (67), the integration over α in (65) is performed when $q^{-r} \le |\alpha| \le q^r|2 + \theta z|^{-1}$. The integral (65) is equal to

$$\iiint \int_{q^{-r} \le |\alpha| \le q^r |2+\theta z|^{-1}} |\alpha|^3 \psi(\alpha(x_1 + 2\theta(x_3 - 1))\chi(\alpha) d^\times \alpha dx_1 dx_2 dx_3,$$

where the triple integral is taken over the set defined by

$$|x_1| \le q^r,\ |x_2| \le q^r,\ |x_3| \le q^r,\ |x_1 + 2\theta(x_3 - 1)| \le |2 + \theta z| q^{-r}.$$

Define l by $|2 + \theta z| = q^l$. The integral above can be writen as

$$\sum_{l \le 2r} q^{l-r} \iiint \int_{q^{-r} \le |\alpha| \le q^{r-l}} |\alpha|^3 \psi(\alpha(x_1 + 2\theta(x_3 - 1))\chi(\alpha) d^\times \alpha dt dx_2 dx_3, \tag{68}$$

where (for each l) x_1, x_2 and x_3 range over the set

$$|x_1| \le q^r,\ |x_2| \le q^r,\ |x_3| \le q^r,\ |2 + \theta z| = q^l,\ |x_1 + 2\theta(x_3 - 1)| \le q^{l-r}. \tag{69}$$

Applying (67), we obtain

$$\int_{q^{-r} \le |\alpha| \le q^{r-l}} |\alpha|^3 \psi(\alpha(x_1 + 2\theta(x_3 - 1)))\chi(\alpha) d^\times \alpha = \sum_{k=-r}^{r-l} (q^3X)^k$$

$$= \frac{(q^3X)^{r-l+1} - (q^3X)^{-r}}{q^3X - 1}. \tag{70}$$

Making the change of variables in (69), $x_1 = 2\theta(1-x_3) + \boldsymbol{\pi}^{r-l}t$, where $|t| \le 1$, the sum (68) is equal to

$$\sum_{l \le 2r} \mathrm{vol}(V_1(l)) q^{l-r} \frac{(q^3X)^{r-l+1} - (q^3X)^{-r}}{q^3X - 1}, \tag{71}$$

where $V_1(l)$ is the set defined by

$$|t| \le 1,\ |x_2| \le q^r,\ |x_3| \le q^r,\ |4-\theta^2 - 2\theta\boldsymbol{\pi}^{r-l}tx_3 + (2\theta x_3 - \theta)^2 - \theta x_2^2| = q^l. \tag{72}$$

Note that since $|x_3| \le q^r$ and $|t| \le 1$, we have $|2\theta\boldsymbol{\pi}^{r-l}tx_3| \le q^l$. We distinguish between the following subcases.

Case 1a. Assume that $|2\theta\boldsymbol{\pi}^{r-l}tx_3| = q^l$. It follows that $|t| = 1$ and $|x_3| = q^r$. This subset of (72) is given by

$$|t| = 1,\ |x_2| \le q^r,\ |x_3| = q^r,\ |4-\theta^2 - 2\theta\boldsymbol{\pi}^{r-l}tx_3 + (2\theta x_3 - \theta)^2 - \theta x_2^2| = q^l. \tag{73}$$

Since θ is a non-square element and $|x_2| \le q^r$, $|(2\theta x_3 - \theta)^2 - \theta x_2^2| = q^{2r}$. Thus the only l when the set (73) is non-empty is when $l = 2r$. Once simplified, it is defined by

$$|t| = 1,\ |x_2| \le q^r,\ |x_3| = q^r,\ |2\boldsymbol{\pi}^{-r}tx_3 + x_2^2 - \theta(2x_3)^2| = q^{2r}.$$

Alternatively, once x_2 and x_3 are fixed, t can be any element with $|t| = 1$ which does not belong to $(\theta(2x_3)^2 - x_2^2)\boldsymbol{\pi}^r/(2x_3) + \boldsymbol{\pi}R$. Thus, the volume of (73) is equal to

$$q^{2r}\Big(1 - \frac{1}{q}\Big)\Big(1 - \frac{2}{q}\Big). \tag{74}$$

Case 1b. Assume that $|2\theta\boldsymbol{\pi}^{r-l}tx_3| < q^l$. This subset of (72) is given by

$$|t| \le 1,\ |x_2| \le q^r,\ |x_3| = q^r,\ |tx_3| < q^r,\ |4-\theta^2 + (2\theta x_3 - \theta)^2 - \theta x_2^2| = q^l. \tag{75}$$

Removing the third inequality, we enlarge the set (75) by

$$|t| \le 1,\ |x_2| \le q^r,\ |x_3| = q^r,\ |tx_3| = q^r,\ |4-\theta^2 + (2\theta x_3 - \theta)^2 - \theta x_2^2| = q^l. \tag{76}$$

Note this subset is non-empty only when $l = 2r$, in which case its volume is

$$\int_{|t|=1|} dt \int_{|x_2| \le q^r} dx_2 \int_{|x_3| = q^r} dx_3 = \Big(1 - \frac{1}{q}\Big)^2 q^{2r}. \tag{77}$$

Thus, when $l < 2r$, the set (75) is given by

$$|t| \le 1,\ |x_2| \le q^r,\ |x_3| = q^r,\ |tx_3| < q^r,\ |4-\theta^2 + (2\theta x_3 - \theta)^2 - \theta x_2^2| = q^l, \tag{78}$$

and, when $l = 2r$ it is the difference of (77) and (76).

We obtained that when $l < 2r$, $\mathrm{vol}(V_1(l)) = W_l - W_{l-1}$, and when $l = 2r$,

$$\mathrm{vol}(V_1(2r)) = W_{2r} - W_{2r-1} + (74) - (77).$$

Note that when $l = 2r$, (70) is equal to $(q^3X)^{-r}$ and (74)$-$(77) is

$$q^{2r}\Big(1 - \frac{1}{q}\Big)\Big(1 - \frac{2}{q}\Big) - \Big(1 - \frac{1}{q}\Big)^2 q^{2r} = -\Big(1 - \frac{1}{q}\Big)\frac{1}{q}q^{2r}.$$

The integral (65) over the subset of Case 1 is equal to

$$\sum_{l \leq 2r}(W_l - W_{l-1})q^{l-r}\frac{(q^3X)^{r-l+1} - (q^3X)^{-r}}{q^3X - 1} - \Big(1 - \frac{1}{q}\Big)\frac{1}{q}X^{-r}, \tag{79}$$

where W_l is the volume of the set defined by

$$|x_2| \leq q^r,\ |x_3| = q^r,\ |4 - \theta^2 + (2\theta x_3 - \theta)^2 - \theta x_2^2| \leq q^l.$$

Case 2. We have $q^r/|2 + \theta z| > 1/|x_1 + 2\theta(x_3 - 1)|$. Using (67), the integration over α in (65) is performed when $q^{-r} \leq |\alpha| \leq q|x_1 + 2\theta(x_3 - 1)|^{-1}$. The integral (65) is equal to

$$\iiint \int_{q^{-r} \leq |\alpha| \leq q|x_1 + 2\theta(x_3 - 1)|^{-1}} |\alpha|^3\psi(\alpha(x_1 + 2\theta(x_3 - 1))\chi(\alpha)d^{\times}\alpha dx_1 dx_2 dx_3,$$

where the triple integral is taken over the set defined by

$$|x_1| \leq q^r,\ |x_2| \leq q^r,\ |x_3| \leq q^r,\ |x_1 + 2\theta(x_3 - 1)| > |2 + \theta z|q^{-r}.$$

Define l by $|x_1 + 2\theta(x_3 - 1)| = q^l$. The integral above can be writen as

$$\sum_{l \leq r} q^{l-r} \iiint \int_{q^{-r} \leq |\alpha| \leq q|x_1 + 2\theta(x_3 - 1)|^{-1}} |\alpha|^3\psi(\alpha(x_1 + 2\theta(x_3 - 1))\chi(\alpha)d^{\times}\alpha dt dx_2 dx_3, \tag{80}$$

where (for each l) x_1, x_2 and x_3 range over the set

$$|x_1| \leq q^r,\ |x_2| \leq q^r,\ |x_3| \leq q^r,\ |x_1 + 2\theta(x_3 - 1)| = q^{,}\ |2 + \theta z| < q^{l+r}. \tag{81}$$

Applying (67), we split the integral over α into two integrals

$$\Big[\int_{q^{-r} \leq |\alpha| \leq q^{-l}} + \int_{|\alpha| = q^{1-l}}\Big]|\alpha|^3\psi(\alpha(x_1 + 2\theta(x_3 - 1)))\chi(\alpha)d^{\times}\alpha = \sum_{k=-r}^{-l}(q^3X)^k - \frac{(q^3X)^{1-l}}{q - 1}$$

$$= \frac{(q^3X)^{1-l} - (q^3X)^{-r}}{q^3X - 1} - \frac{(q^3X)^{1-l}}{q - 1}. \tag{82}$$

Making the change of variables in (81), $x_1 = 2\theta(1-x_3) + \boldsymbol{\pi}^{-l}\epsilon$, where $|\epsilon| = 1$, the sum (80) is equal to

$$\sum_{l \le r} \mathrm{vol}(V_2(l)) q^l \Big[\frac{(q^3X)^{1-l} - (q^3X)^{-r}}{q^3X - 1} - \frac{(q^3X)^{1-l}}{q-1}\Big], \tag{83}$$

where $V_2(l)$ is the set defined by

$$|\epsilon| = 1,\ |x_2| \le q^r,\ |x_3| \le q^r,\ |4 - \theta^2 - 2\theta\boldsymbol{\pi}^{r-l}\epsilon x_3 + (2\theta x_3 - \theta)^2 - \theta x_2^2| = q^l. \tag{84}$$

Note that since $|x_3| \le q^r$ and $|\epsilon| = 1$, we have $|2\theta\boldsymbol{\pi}^{-l}\epsilon x_3| \le q^{r+l}$. We distinguish between the following subcases.

Case 2a. Assume that $|2\theta\boldsymbol{\pi}^{-l}\epsilon x_3| = q^{l+r}$. It implies that $|x_3| = q^r$. Following the same argument as in Case 1a, we conclude that with this assumption, the only non-empty subset of (84) is when $l = r$. It is defined by

$$|\epsilon| = 1,\ |x_2| \le q^r,\ |x_3| = q^r,\ |2\boldsymbol{\pi}^{-r}\epsilon x_3 + x_2^2 - \theta(2x_3)^2| \le q^{2r-1}. \tag{85}$$

Alternatively, once x_2 and x_3 are fixed, ϵ should be in $-(x_2^2 - \theta(2x_3)^2)\boldsymbol{\pi}^r/(2x_3) + \boldsymbol{\pi}R$. Note that $|\epsilon| = 1$. The volume of (85) is

$$q^{2r}\Big(1 - \frac{1}{q}\Big)\frac{2}{q}.$$

When $l = r$ (82) is equal to $(q^3X)^{-r} - (q^3X)^{1-r}/(q-1)$. The contribution of this case to (83) is

$$q^{2r}\Big(1 - \frac{1}{q}\Big)\frac{2}{q}q^r\Big[(q^3X)^{-r} - \frac{(q^3X)^{1-r}}{q-1}\Big] \tag{86}$$

Case 2b. Assume that $|2\theta\boldsymbol{\pi}^{-l}\epsilon x_3| \le q^{l+r-1}$. Thus this subset of (84) is given by

$$|\epsilon| = 1,\ |x_2| \le q^r,\ |x_3| = q^r,\ |4 - \theta^2 + (2\theta x_3 - \theta)^2 - \theta x_2^2| \le q^{l+r-1}. \tag{87}$$

Note that the volume of this set is $\Big(1 - \frac{1}{q}\Big)W_{l+r-1}$.

Combining these two subcases, the integral (65) over the subset of Case 2 is equal to

$$\sum_{l \le r} W_{l+r-1} q^l \Big(1 - \frac{1}{q}\Big)\Big[\frac{(q^3X)^{1-l} - (q^3X)^{-r}}{q^3X - 1} - \frac{(q^3X)^{1-l}}{q-1}\Big] + \frac{1}{q}\Big(1 - \frac{1}{q}\Big)X^{-r} - qX^{1-r}. \tag{88}$$

The answer is obtained on adding (79) to (88). Fix any $k < 2r$. To find the coefficient of W_k in (79), we consider the terms when $l = k$ and $l = k+1$. This coefficient is equal to

$$q^{k-r}\frac{(q^3X)^{r-k+1} - (q^3X)^{-r}}{q^3X - 1} - q^{k+1-r}\frac{(q^3X)^{r-k} - (q^3X)^{-r}}{q^3X - 1}. \tag{89}$$

Similarly, to find the coefficient of W_k in (88), we consider the term with $l = 1 + k - r$. The coefficient is equal to

$$\Big(1 - \frac{1}{q}\Big) q^{1+k-r} \left[\frac{(q^3X)^{r-k} - (q^3X)^{-r}}{q^3X - 1} - \frac{(q^3X)^{r-k}}{q-1} \right]$$

$$= q^{k+1-r} \frac{(q^3X)^{r-k} - (q^3X)^{-r}}{q^3X - 1} - q^{k-r} \frac{(q^3X)^{r-k} - (q^3X)^{-r}}{q^3X - 1} - q^{k-r}(q^3X)^{r-k}. \tag{90}$$

The second term of (89) cancels the first one of (90). Thus, their sum is zero. Futher, note that

$$W_{2r} = \int_{|x_2| \le q^r} dx_2 \int_{|x_3| = q^r} dx_3 = \Big(1 - \frac{1}{q}\Big) q^{2r}.$$

Thus the sum of (79) and (88) is equal to

$$W_{2r} q^r \frac{(q^3X)^{1-r} - (q^3X)^{-r}}{q^3X - 1} - \Big(1 - \frac{1}{q}\Big)\frac{1}{q} X^{-r} + \Big(1 - \frac{1}{q}\Big)\frac{1}{q} X^{-r} - qX^{1-r} = X^{-r} - qX^{1-r}.$$

The Proposition is proved. $\square$

Theorem. *Corresponding f and f' are matching.*

Proof. Indeed as we have seen in Section I.0, to prove that corresponding functions are matching (i.e. $\Psi(\alpha, \phi_f) = \psi(\alpha)|\alpha|\Psi'(\alpha^{-1}, \phi'_{f'})$) it is enough to show that (for $r \ge 0$)

$$\Psi(\alpha, \Phi_r) = \psi(\alpha)|\alpha|\Psi'(\alpha^{-1}, (-1)^r q^r(\phi'_r \pm \phi'_{r-1})). \tag{91}$$

Comparing Proposition 3.5 in the split case, and Proposition 3.6 in the non-split case, with Proposition 3.1, we have (for $r \ge 1$)

$$\int_{F^\times} \Psi(\alpha, \Phi_r)\chi(\alpha) d^\times\alpha = \int_{F^\times} \psi(\alpha)|\alpha|\Psi'(\alpha^{-1}, (-1)^r q^r(\phi'_r \pm \phi'_{r-1}))\chi(\alpha) d^\times\alpha,$$

where χ is any complex valued character of $F^\times$. If χ is ramified both integrals are equal to 0. Fourier inversion formula now implies (91) when $r \ge 1$. When $r = 0$, the formula (91) follows from the unit element case, treated in [FM]. $\square$

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